Electrodynamics of Topological Insulators

Author:
MICHAEL SAMMON

Advisor:
PROFESSOR HARSH MATHUR

DEPARTMENT OF PHYSICS
CASE WESTERN RESERVE UNIVERSITY
CLEVELAND, OH 44106-7079
May 2, 2014
0.1 Abstract

Topological insulators are new metamaterials that have an insulating interior, but a conductive surface. The specific nature of this conducting surface causes a mixing of the electric and magnetic fields around these materials. This project investigates this effect to deepen our understanding of the electrodynamics of topological insulators.

The first part of the project focuses on the fields that arise when a current carrying wire is brought near a topological insulator slab and a cylindrical topological insulator. In both problems, the method of images was able to be used. The result were image electric and magnetic currents. These magnetic currents provide the manifestation of an effect predicted by Edward Witten for Axion particles. Though the physics is extremely different, the overall result is the same in which fields that seem to be generated by magnetic currents exist. The second part of the project begins an analysis of a topological insulator in constant electric and magnetic fields. It was found that both fields generate electric and magnetic dipole like responses from the topological insulator, however the electric field response within the material that arise from the applied fields align in opposite directions. Further investigation into the effect this has, as well as the overall force that the topological insulator experiences in these fields will be investigated this summer.
List of Figures

0.4.1 Dyon\(^1\) fields of a point charge near a Topological Insulator ........................................ 6
B..1 Diagram of a Wire of Current I near a Topological Insulator ........................................ 18
C..1 Diagram of a Wire and Image Currents for a Cylindrical Topological Insulator ........ 21
D..1 Spherical Topological Insulator in a Constant E field ........................................... 25
Contents

0.1 Abstract ................................................................. 1
0.2 Background and Motivation ........................................ 4
0.3 Purpose and Objectives .............................................. 5
0.4 Zhang's Calculations ............................................... 6
0.5 Current Carrying Wires and Topological Insulators .......... 8
0.6 Repulsion, Force Analysis, and Spherical Topological Insulators .... 10
0.7 Conclusion and Beyond Senior Project .......................... 11
0.8 Appendix .................................................................. 12
   A. Modified Maxwell’s Equations and Boundary Conditions .......... 12
   B. Boundary Conditions within Materials, Monopole Currents, and Fields of a Wire near a Topological Insulator Slab ................. 16
   C. Wire Near a Cylindrical Topological Insulator .................. 21
   D. Spherical Topological Insulator in Constant Electric and Magnetic Fields  . 25
0.2 Background and Motivation

The electrodynamics of a typical insulator are well understood. Electric fields induce an electric polarization of the material, while magnetic fields induce a magnetic polarization. In most practical cases, this effect can be well approximated by a linear response described by the dielectric constant $\epsilon$ and the magnetic permeability $\mu$. This linear response is fully described by the electromagnetic action

$$S_0 \equiv \int dt \int d^3x \frac{\epsilon_0}{2} \left( \nabla \phi + \frac{\partial A}{\partial t} \right)^2 - \frac{1}{2\mu_0} (\nabla \times A)^2 - \rho \phi + j \cdot A,$$

which under functional variation will give the standard set of Maxwell’s equation. These equations can then be generalized to describe polarized materials by separating the charge density into bound and free charge, and rewriting the equations in terms of the displacement field $D$. The details of this are found in Appendix B.

In a recent development, new materials known as topological insulators have been shown to follow a different set of Maxwell’s equations. These topological insulators\(^1\) are insulators which allow conducting surface states that exhibit the Quantum Hall Effect. The specifics of this effect are outlined in Zhang\(^1\) and will not be discussed. What is important is that the effect of this conducting surface can be describe by the addition of the magneto-electric action

$$S_\theta = -\theta \sqrt{\frac{\epsilon_0}{\mu_0}} (\nabla \phi + \frac{\partial A}{\partial t}) \cdot (\nabla \times A)$$

(0.2.2)

to the typical electromagnetic action\(^1\). Here $\theta$ is a parameter that describes the topological insulator. The physical result is that an applied magnetic field will induce a surface charge, which will in turn cause an electric polarization.

The effect of this magneto-electric action is described by an alteration to Maxwell’s equations

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} - c \nabla \theta \cdot B$$

(0.2.3)

$$\nabla \cdot B = 0$$

(0.2.4)
\[ \nabla \times E = -\frac{\partial B}{\partial t} \]  
\[
\nabla \times B = \mu_0 j + \frac{1}{c^2} \frac{\partial E}{\partial t} + \frac{1}{c} (B \frac{\partial \theta}{\partial t} + \nabla \theta \times E) \]  
(0.2.5)  

\[ \nabla \times E = -\frac{\partial B}{\partial t} \]  
\[ \nabla \times B = \mu_0 j + \frac{1}{c^2} \frac{\partial E}{\partial t} + \frac{1}{c} (B \frac{\partial \theta}{\partial t} + \nabla \theta \times E) \]  
(0.2.6)

, in which the electric and magnetic fields are coupled by the parameter theta. A derivation of these laws is found in Appendix A. As Wilczek\(^2\) showed, these equations cause a manifestation of the Witten\(^3\) effect, in which a magnetic monopole image charge is created by an electric point charge. Though the original calculation was describing the electrodynamic effect of an axion domain wall, topological insulators follow the same magneto-electric action, and therefore provide a physical manifestation of the Witten effect.

### 0.3 Purpose and Objectives

The purpose of this project is to further understand the electrodynamics of these materials. Topological insulators are relatively new, and though there has been extensive study into the quantum properties that these materials exhibit, there has been little inquiry into their electrodynamic property.

The project began with two basic objectives.

1) Calculate the electromagnetic response of a topological insulator slab when a current carrying wire is brought near.

2) Calculate the electromagnetic response of a cylindrical topological insulator when a current carrying wire is brought near.

Zhang’s point charge calculations provided a basis for understanding the electrostatic interaction for a topological insulator. Our project provides a basis for the magnetostatic interaction.
0.4 Zhang’s Calculations

Zhang’s paper described the basic interaction between a point charge and a topological insulator slab. Appendix A and B outline the necessary boundary conditions that describe a topological insulator.

Figure 0.4.1: Dyon fields of a point charge near a Topological Insulator

For the simple case of a point charge near a topological insulator slab, the method of images may be used to calculate the resulting fields. As Zhang showed, the scenario in which a point charge in air is brought near a topological insulator with dielectric constant $\epsilon_r$ and magnetic permeability $\mu_r$, can be described by two image dyons; one located at the location of the point charge and one reflected across the boundary of the insulator. The values of the charges are given by:

\[
q_{\text{outside}} = \frac{1 - \epsilon_r - \frac{\mu_r}{1 + \mu_r} \theta^2 q}{1 + \epsilon_r + \frac{\mu_r}{1 + \mu_r} \theta^2 q} \quad (0.4.1)
\]

\[
q_{\text{inside}} = \frac{2\epsilon_r}{1 + \epsilon_r + \frac{\mu_r}{1 + \mu_r} \theta^2 q} \quad (0.4.2)
\]

\[
g_{\text{inside}} = \frac{-2\frac{\mu_r}{1 + \mu_r} \theta c}{1 + \epsilon_r + \frac{\mu_r}{1 + \mu_r} \theta^2 q} \quad (0.4.3)
\]

\[
g_{\text{outside}} = \frac{2\theta c}{1 + \epsilon_r + \frac{\mu_r}{1 + \mu_r} \theta^2 q} \quad (0.4.4)
\]

Here $g$ represents the magnetic charge, while $q$ is used to label the electric charge. As we see, all
the image charges are proportional to the original charge \( q \). We also find that both the magnetic charges, and the magnetic permeability terms are proportional to the topological parameter \( \theta \), so that in the limiting case that \( \theta \to 0 \), we are left with the typical dielectric image charges. The resulting fields and the directions can be found in Figure 0.4.1. It is best to clarify that these image monopoles originate from the induced surface currents on the topological insulator rather than physical monopoles.

The existence of monopoles may be a frightening thought. It seems to violate Maxwell’s second law, which remained unaltered in the case of topological insulators. The real statement of this law is that the magnetic flux through any closed surface must be zero, which can usually be expressed as a statement of the divergence of the B field when applying Gauss’s law. In order to determine the validity of the second law, Zhang calculated the interaction between a point charge and a spherical topological insulator. In this case, in addition to a monopole at the typical image location for spherical objects, an image line of magnetic charge\(^1\) extends from the center of the sphere to the image charge is generated. This line charge exactly cancels the magnetic flux in the sphere, proving that Maxwell’s laws are still in tact.

Zhang et al. then tried to propose a possible experiment to measure the Witten effect. As it turns out, if the distance between the point charge and the sphere is significantly smaller than the radius of the sphere, the image monopole charge dominates over the magnetic charge density. Therefore, the Witten effect can be realized in a spherical topological insulator by utilizing a magnetic force microscope. In order to measure the resulting charge, interference from the electric image charge, as well as defects in the fields resulting from impurities in the surface must be taken into account. These interferences force extreme precision in the necessary measurement and have yet to be measured.
0.5 Current Carrying Wires and Topological Insulators

The first problem solved was the case of a wire of constant current near a topological insulator slab. For the purpose of generality, the topological insulator was assumed to have relative permittivity \( \varepsilon_r \) and permeability \( \mu_r \). The similarity between the point charge problem and this problem implied that the method of images may be used to determine the fields. Indeed, by assuming both electric and magnetic currents, the fields everywhere can be calculated. The values of these currents are given by:

\[
I' = \frac{(\mu_r - 1 - \frac{\mu_r \theta_0^2}{\varepsilon_r + 1})}{(\mu_r + 1 + \frac{\mu_r \theta_0^2}{\varepsilon_r + 1})} I
\]

\[
I'' = \frac{2}{(\mu_r + 1 + \frac{\mu_r \theta_0^2}{\varepsilon_r + 1})} I
\]

\[
J' = - \frac{c \theta_0 \mu_r}{(\varepsilon_r + 1)(\mu_r + 1 + \frac{m u_r \theta_0^2}{\varepsilon_r + 1})} I
\]

\[
J'' = \frac{c \theta_0 \mu_r \varepsilon_r}{(\varepsilon_r + 1)(\mu_r + 1 + \frac{m u_r \theta_0^2}{\varepsilon_r + 1})} I.
\]

Here \( I' \) and \( J' \) describe the image electric and magnetic currents that lie within the slab respectively. Similarly, \( I'' \) and \( J'' \) describe the image currents that lie outside of the slab and determine the fields within it. The details of this calculation are given in Appendix B.

It is important to highlight some features of these currents. First and foremost, if \( \theta \to 0 \), the magnetic currents disappear, and the electric currents reduce to

\[
I' = \frac{(\mu_r - 1)}{(\mu_r + 1)} I
\]

\[
I'' = \frac{2}{(\mu_r + 1)} I,
\]

which are the fields one would find for a typical insulator with relative permeability \( /mu_r \). Another important feature is brought to light when the insulator has no polarization or magnetization. We
set $\mu_r$ and $\epsilon_r$ equal to one in this limit, and find that the image currents take the form

$$I' = -\frac{\theta_0^2}{4} I (0.5.7)$$

$$I'' = \frac{1}{1 + \frac{\theta_0^2}{4}} I (0.5.8)$$

$$J' = -\frac{\theta_0}{2} \frac{\theta_0}{4} I (0.5.9)$$

$$J'' = \frac{\theta_0}{2} \frac{\theta_0}{4} I. (0.5.10)$$

Note the minus sign of $I'$. It is known that currents flowing in the opposite direction repel each other. Indeed, by calculating the stress tensor of these fields, the force per unit length between the insulator and the wire can be found to be

$$F_x = \frac{\mu_0 \theta_0^2 I^2}{16\pi d} \frac{1}{1 + \frac{\theta_0^2}{4}}. (0.5.11)$$

The positive sign of this force shows that the objects repel each other. This idea of repulsion leads into the second part of the project in which the stability of this force is brought into question.

The second problem that was solved was the case of a wire near a cylindrical topological insulator. In a surprising, yet delightful twist, this problem can be solved using the method of images by including electric and magnetic image currents at the center of the cylinder. The details of this calculation are given in Appendix C. The currents very much resemble the values found for the topological insulator slab with no polarization nor magnetization, leading to the question of the nature of the similarity. Professor Mathur suggests that there may be a conformal mapping between the two problems, but we have yet to find such a result.
0.6 Repulsion, Force Analysis, and Spherical Topological Insulators

As the previous section showed, a topological insulator brought near a magnetic wire experienced a repulsive force. Magnetic repulsion is a very interesting property. As Berry and Geim\textsuperscript{4} showed, diamagnetic materials can achieve stable levitation. We wanted to investigate whether a topological insulator could also achieve stable levitation. In order to replicate the analysis that was done by Berry and Geim, a theory of the forces experienced by the topological insulator needs to be developed.

We wanted to start by considering the simplest case, a topological insulator in constant fields. Three cases have been considered so far: a constant electric field, a constant magnetic field, and a constant electric and magnetic field that points in the same direction. Appendix D details these calculations, and includes the resulting fields. In all three cases, a dipole like response occurs within the topological insulator, with some interesting properties. The electric field that is generated within the TI by the constant magnetic field opposes the direction of the field, while the field response of the TI exposed to the Electric field is in the same direction of the applied field. For the case in which both a constant electric and magnetic field are applied in the same direction, the terms compete against one another, as eq. D.24 shows. If the fields are chosen so that \( \frac{E_0}{B_0} = \frac{\theta c}{3} \), the electric potential (and the field) completely disappear within the insulator. This has not been investigated in detail, but could prove to have interesting results when calculating the force experienced by these fields.

Unfortunately, the semester ended before I could calculate the forces for these cases, but one thing can be noted. The stress tensor depends on the square of the fields. In the case in which the electric and magnetic field are both applied, there are terms in the potential proportional to each field. This indicates that the stress tensor, and as a result the force, will have a cross term that would not appear in the cases separately.
0.7 Conclusion and Beyond Senior Project

The first part of the project showed several examples of the Witten Effect\(^1\) beyond those illustrated by Zhang. If the resulting tangential electric fields could be measured, it will provide the first experimental evidence of this phenomena. As for the second part of the project, I plan to continue working with Professor Mathur to finish analyzing the forces that a spherical TI experiences when immersed in constant electric and magnetic fields. I also plan to generalize the third case to electric and magnetic fields oriented arbitrarily to highlight any interesting effect that the orientation could have on the forces involved. I hope that these calculations will provide enough material to replicate Berry and Geim’s analysis and determine the stability of the repulsion, and possibly provide another means of magnetic levitation.
A. Modified Maxwell’s Equations and Boundary Conditions

Here we will derive the boundary conditions that completely describe the electrodynamics of topological insulators. This is accomplished by varying the action

\[ S = S_0 + S_\theta \]  

where

\[ S_0 \equiv \int dt \int d^3x \frac{\epsilon}{2} (\nabla \phi + \frac{\partial A}{\partial t})^2 - \frac{1}{2\mu_0} (\nabla \times A)^2 - \rho \phi + j \cdot A \]

is the typical electromagnetic action and

\[ S_\theta = -\theta \sqrt{\frac{\epsilon}{\mu_0}} (\nabla \phi + \frac{\partial A}{\partial t}) \cdot (\nabla \times A) \]

is an addition that describes the physics of the topological insulator. Here \( \theta \) is a parameter determined by the topological insulator. By extremizing the action, we can derive the set of laws that completely govern the electrodynamics of these objects.

We begin by varying the scalar potential. This is achieved by letting

\[ \phi \rightarrow \phi + \delta \phi \]

where \( \delta \phi \) is infinitesimal and zero on the surface of integration. We then take the difference

\[ S[\phi + \delta \phi] - S[\phi] = \delta S \]

and set it equal to zero. Now it is easy to see that

\[ \delta S = \delta S_0 + \delta S_\theta \]
and we know that
\[ \delta S_0 = \int dt \int d^3x ( - (\epsilon_0 \nabla \cdot (\nabla \phi + \frac{\partial A}{\partial t}) + \rho) \delta \phi) \]  
so we need only determine \( \delta S_\theta \). We see to first order in \( \delta \phi \)
\[ S_\theta[\phi + \delta \phi] = \int dt \int d^3x ( -\theta \sqrt{\frac{\epsilon_0}{\mu_0}} (\nabla \phi + \frac{\partial A}{\partial t}) \cdot (\nabla \times A) - \theta \sqrt{\frac{\epsilon_0}{\mu_0}} (\nabla \times A) \cdot \nabla \delta \phi) \]
\[ = S_\theta[\phi] - \int dt \int d^3 \theta \sqrt{\frac{\epsilon_0}{\mu_0}} (\nabla \times A) \cdot \nabla \delta \phi \]
from which it follows that
\[ \delta S_\theta = \int dt \int d^3x ( -\theta \sqrt{\frac{\epsilon_0}{\mu_0}} (\nabla \times A) \cdot \nabla \delta \phi) \]
. In order to put this into a useful form, we integrate by parts, so that
\[ \delta S_\theta = \int dt \int d^3x (\nabla \cdot (\theta \sqrt{\frac{\epsilon_0}{\mu_0}} (\nabla \times A) \delta \phi) \]
where I have ignored the surface term because \( \delta \phi \) is zero on the surface. Utilizing the fact that
the divergence of a curl is zero, we find that
\[ \delta S_\theta = \int dt \int d^3x (\nabla \cdot (\theta \sqrt{\frac{\epsilon_0}{\mu_0}} (\nabla \times A) \delta \phi) \]
which we can add to the variation of the typical electromagnetic action to find
\[ \delta S = \int dt \int d^3x ((\nabla \cdot (\theta \sqrt{\frac{\epsilon_0}{\mu_0}} (\nabla \times A) - (\epsilon_0 \nabla \cdot (\nabla \phi + \frac{\partial A}{\partial t}) + \rho) \delta \phi \]
. \( \delta \phi \) was arbitrary, so the requirement that \( \delta S = 0 \) forces the term in parentheses to be zero.
Therefore,
\[ ((\nabla \cdot (\theta \sqrt{\frac{\epsilon_0}{\mu_0}} (\nabla \times A) - (\epsilon_0 \nabla \cdot (\nabla \phi + \frac{\partial A}{\partial t}) + \rho)) = 0 \]
which after rearranging and dividing by $\epsilon_0$ we arrive at

$$-\nabla \cdot (\nabla \phi + \frac{\partial A}{\partial t}) = \frac{\rho}{\epsilon_0} - c\nabla \theta \cdot (\nabla \times A)$$ (A.15)

which is a modified version of Gauss’s law.

Varying the vector potential is messier, but follows the exact same procedure. We begin by expanding $S_\theta[A + \delta A]$ to first order in $\delta A$. To first order, we find that

$$S_\theta[A + \delta A] = S_\theta[A] - \int dt \int d^3x \theta \sqrt{\frac{\epsilon_0}{\mu_0}} \left( \frac{\partial A}{\partial t} \cdot (\nabla \times A) + (\nabla \phi + \frac{\partial A}{\partial t}) \cdot (\nabla \times A) \right)$$ (A.16)

from which it follows that

$$\delta S_\theta = -\int dt \int d^3x \theta \sqrt{\frac{\epsilon_0}{\mu_0}} \left( \frac{\partial A}{\partial t} \cdot (\nabla \times A) + (\nabla \phi + \frac{\partial A}{\partial t}) \cdot (\nabla \times A) \right)$$ (A.17)

. Again, to put this into a useful form, we integrate by parts. Ignoring surface terms, we find that

$$\delta S_\theta = \int dt \int d^3x \sqrt{\frac{\epsilon_0}{\mu_0}} \left( \frac{\partial \theta}{\partial t} \cdot (\nabla \times A) - \nabla \cdot (\nabla \theta \times (\nabla \phi + \frac{\partial A}{\partial t})) \right) \delta A$$ (A.18)

which can be simplified to

$$\delta S_\theta = \int dt \int d^3x \sqrt{\frac{\epsilon_0}{\mu_0}} \left( \frac{\partial \theta}{\partial t} \cdot (\nabla \times A) - \nabla \cdot (\nabla \theta \times (\nabla \phi + \frac{\partial A}{\partial t})) \right) \delta A$$ (A.19)

. Adding this to the variation of $\delta S_0$, we can arrive at the total variation

$$\int dt \int d^3x \left( -\epsilon_0 \frac{\partial (\nabla \phi + \frac{\partial A}{\partial t})}{\partial t} - \frac{1}{\mu_0} \nabla \times (\nabla \times A) + \frac{\epsilon_0}{\mu_0} \left( \frac{\partial \theta}{\partial t} \cdot (\nabla \times A) - \nabla \cdot (\nabla \theta \times (\nabla \phi + \frac{\partial A}{\partial t})) \right) \right) \delta A$$ (A.20)

Again $\delta A$ is arbitrary, so in order for $\delta S = 0$ we require the term in parentheses to be zero. Rearranging and multiplying by $\mu_0$, we find that

$$\nabla \times (\nabla \times A) = \mu_0 j - \frac{1}{c^2} \frac{\partial (\nabla \phi + \frac{\partial A}{\partial t})}{\partial t} + \frac{1}{c} \frac{\partial \theta}{\partial t} \left( \nabla \times A - \nabla \theta \times (\nabla \phi + \frac{\partial A}{\partial t}) \right)$$ (A.21)
which is a modified version of Ampere’s law. The other laws can be derived from these. Utilizing the definitions of \( E \) and \( B \), we can now write down the modified form of Maxwell’s laws:

\[
\nabla \cdot E = \frac{\rho}{\epsilon_0} - c\nabla \theta \cdot B \tag{A..22}
\]

\[
\nabla \cdot B = 0 \tag{A..23}
\]

\[
\nabla \times E = -\frac{\partial B}{\partial t} \tag{A..24}
\]

\[
\nabla \times B = \mu_0 j + \frac{1}{c^2} \frac{\partial E}{\partial t} + \frac{1}{c} (B \frac{\partial \theta}{\partial t} + \nabla \theta \times E) \tag{A..25}
\]

In order to derive the boundary conditions for a topological insulator, we consider the case of static fields. In this scenario, \( \frac{\partial E}{\partial t} = 0, \frac{\partial B}{\partial t} = 0, \) and \( \frac{\partial \theta}{\partial t} = 0 \). The modified laws then read

\[
\nabla \cdot E = \frac{\rho}{\epsilon_0} - c\nabla \theta \cdot B \tag{A..26}
\]

\[
\nabla \cdot B = 0 \tag{A..27}
\]

\[
\nabla \times E = 0 \tag{A..28}
\]

\[
\nabla \times B = \mu_0 j + \frac{1}{c} \nabla \theta \times E \tag{A..29}
\]

A topological insulator can be modeled by letting the function \( \theta(x) \) be zero outside of the topological insulator, and constant inside of it. We can then integrate across the boundary to find that

\[
E_{in}^\perp - E_{out}^\perp = -c\theta_0 B_{\perp} \bigg|_{\text{boundary}} \tag{A..30}
\]

\[
B_{\perp \text{continuous}} \tag{A..31}
\]

\[
E_{\parallel \text{continuous}} \tag{A..32}
\]

\[
B_{in}^\parallel - B_{out}^\parallel = \frac{\theta_0}{c} E_{\parallel} \bigg|_{\text{boundary}} \tag{A..33}
\]

which completely describe the dynamics of the topological insulator.
B. Boundary Conditions within Materials, Monopole Currents, and Fields of a Wire near a Topological Insulator Slab

Appendix A outlined the set of boundary conditions that describe the dynamics of a topological insulator. However, for typical materials, it is preferred to have a set of boundary conditions that include permittivity and permeability. To do this, we first note that A.22 can be written as

$$\nabla \cdot (\varepsilon_0 E + \varepsilon_0 c_0 B) = \rho_{total}. \quad (B.1)$$

If the topological insulator is polarizable, then we can separate the charge density

$$\rho_{total} = \rho_{bound} + \rho_{ext} \quad (B.2)$$

where

$$\rho_{bound} = -\nabla \cdot P \quad (B.3)$$

and P is the polarization density of the material. We can modify B.1 to become

$$\nabla \cdot (\varepsilon_0 E + P + \varepsilon_0 c_0 B) = \rho_{ext}. \quad (B.4)$$

We define the displacement field D in the usual way

$$D = \varepsilon_0 E + P. \quad (B.5)$$

Recall that $\nabla \cdot B$ is only nonzero at the boundary for a topological insulator. Then it is true that

$$\nabla \cdot D = \rho_{ext} \quad (B.6)$$

with the following boundary conditions

$$E_{\parallel \text{continuous}} \quad (B.7)$$
\[ D^{out}_\perp - D^{in}_\perp = \epsilon_0 c \theta_0 B_\perp. \]  
\[ (B..8) \]

We can follow the same process for A..25 to find that the auxiliary field \( H \) defined by
\[
H = \frac{B}{\mu_0} - M
\]
\[ (B..9) \]
satisfies
\[
\nabla \times H = j_{\text{ext}}
\]
\[ (B..10) \]
and has the following boundary conditions

\[ B_\perp \text{continuous} \]  
\[ (B..11) \]
\[ H^{out}_\parallel - H^{in}_\parallel = -\frac{\theta_0}{\mu_0 c} E_\parallel. \]  
\[ (B..12) \]

In order to solve the problem of a wire near a topological insulator slab, a brief note must be made of maxwell’s laws with magnetic charges. Suppose that there exist magnetic charges. Then Maxwell’s second law reads
\[
\nabla \cdot B = \mu_0 \rho_m
\]
\[ (B..13) \]
where \( \rho_m \) is the magnetic charge density. As electric currents generate magnetic fields, it is expected that magnetic currents generate electric fields. This changes Faraday’s law to
\[
\nabla \times E = -\frac{\partial B}{\partial t} \pm \mu_0 j_m
\]
\[ (B..14) \]
where \( j_m \) is the magnetic current density. The sign of the additional term must be such that current is conserved. To make this rigorous, we take the divergence of B..14. The divergence of a curl is zero so that we are left with,
\[
0 = \nabla \cdot (-\frac{\partial B}{\partial t} \pm \mu_0 j_m)
\]
\[ (B..15) \]
\[ \Rightarrow 0 = -\frac{\partial \nabla \cdot B}{\partial t} \pm \mu_0 \nabla \cdot j_m \]  
\[ (B..16) \]
\[ \Rightarrow 0 = -\frac{\partial \rho_m}{\partial t} \pm \nabla \cdot j_m. \]  (B.17)

It is clear then that in order for the monopole current to be conserved, we take the minus sign so that Faraday's law becomes

\[ \nabla \times E = -\left( \frac{\partial B}{\partial t} + \mu_0 j_m \right). \]  (B.18)

The minus sign indicates that a positive monopole current will cause an electric field that spirals in the opposite direction of a magnetic field generated by a positive current. Consider the case of a wire with current I located a distance d from an infinite topological insulator slab with relative permittivity \( \epsilon_r \) and relative permeability \( \mu_r \). A diagram of the problem is shown in figure B.1.

The problem can easily be solved using the method of images with electric and magnetic currents. The idea is simple. To determine the field outside, we suppose that in addition to the wire, there is an image wire with electric current I' and magnetic current J' located a distance d within the material. Then to determine the field inside, we suppose there is an electric current I'' and magnetic current J'' a distance d outside the material. Then we may write the field outside of the insulator as

\[ B_x = -\frac{\mu_0 I}{2\pi \left( (x + d)^2 + y^2 \right)} - \frac{\mu_0 I'}{2\pi \left( (x - d)^2 + y^2 \right)} \]  (B.19)

Figure B.1: Diagram of a Wire of Current I near a Topological Insulator
\[ B_y = \frac{\mu_0 I}{2\pi} \frac{(x + d)}{((x + d)^2 + y^2)} + \frac{\mu_0 I'}{2\pi} \frac{(x + d)}{((x - d)^2 + y^2)} \]  
\( B_x = \frac{\mu_0 J'}{2\pi} \frac{y}{((x - d)^2 + y^2)} \)  
\[ E_x = -\frac{\mu_0 J'}{2\pi} \frac{(x - d)}{((x - d)^2 + y^2)} \]  
\[ E_y = \mu_0 J'' \frac{(x - d)}{2\pi} \frac{(x - d)}{((x - d)^2 + y^2)} \]

while the fields inside the insulator are given by

\[ B_x = -\frac{\mu_0 \mu_r I''}{2\pi} \frac{y}{((x + d)^2 + y^2)} \]  
\[ B_y = \frac{\mu_0 \mu_r I''}{2\pi} \frac{(x + d)}{((x + d)^2 + y^2)} \]  
\[ E_x = \frac{\mu_0 J''}{2\pi \epsilon_r} \frac{y}{((x + d)^2 + y^2)} \]  
\[ E_y = -\frac{\mu_0 J''}{2\pi \epsilon_r} \frac{(x + d)}{((x + d)^2 + y^2)}. \]

To determine the values of the image currents, we utilize the boundary conditions. Continuity of \( E_y \) gives

\[ -\frac{\mu_0 J'}{2\pi} \frac{(-d)}{(d^2 + y^2)} = -\frac{\mu_0 J''}{2\pi \epsilon_r} \frac{(d)}{(d^2 + y^2)} \]  
which after canceling like terms leaves

\[ J' = -\frac{J''}{\epsilon_r} \]

while the continuity of \( B_x \) gives

\[ I + I' = \mu_r I''. \]

Utilizing B..8, we find that

\[ J'' - J' = c\theta_0 I'' \mu_r \]  
while B..12 gives

\[ I'' - I + I' = -\frac{\theta_0}{\epsilon \epsilon_r} J''. \]
After solving these four equations we find

\[ I' = \frac{(\mu_r - 1 - \frac{\mu_r \theta_0^2}{\epsilon_r + 1})}{(\mu_r + 1 + \frac{\mu_r \theta_0^2}{\epsilon_r + 1})} I \]  
\[ (B.32) \]

\[ I'' = \frac{2}{(\mu_r + 1 + \frac{\mu_r \theta_0^2}{\epsilon_r + 1})} I \]  
\[ (B.33) \]

\[ J' = -\frac{c \theta_0 \mu_r}{(\epsilon_r + 1)} \frac{2}{(\mu_r + 1 + \frac{\mu_r \theta_0^2}{\epsilon_r + 1})} I \]  
\[ (B.34) \]

\[ J'' = \frac{c \theta_0 \mu_r \epsilon_r}{(\epsilon_r + 1)} \frac{2}{(\mu_r + 1 + \frac{\mu_r \theta_0^2}{\epsilon_r + 1})} I. \]  
\[ (B.35) \]

Note that in the case of no polarization and no magnetization, we find that currents reduce to

\[ I' = -\frac{\theta_0^2}{4} \frac{1}{(1 + \frac{\theta_0^2}{4})} I \]  
\[ (B.36) \]

\[ I'' = \frac{1}{(1 + \frac{\theta_0^2}{4})} I \]  
\[ (B.37) \]

\[ J' = -\frac{\theta_0 c}{2} \frac{1}{(1 + \frac{\theta_0^2}{4})} I \]  
\[ (B.38) \]

\[ J'' = \frac{\theta_0}{2} \frac{1}{(1 + \frac{\theta_0^2}{4})} I \]  
\[ (B.39) \]
C. Wire Near a Cylindrical Topological Insulator

Here the fields of a wire of current $I$ near a cylindrical topological insulator with no relative permittivity or permeability is solved. As Zhang\textsuperscript{1} showed, the problem of a point charge near a spherical topological insulator generated fields that acted as if they were sourced from lines of monopole and electric charge, and could not be solved using the method of images. Despite this, a wire near a topological insulator of this geometry can be reduced an image problem by including currents at the center as well as the classical image point.

![Diagram of a Wire and Image Currents for a Cylindrical Topological Insulator](image)

Figure C.1: Diagram of a Wire and Image Currents for a Cylindrical Topological Insulator

To calculate the fields outside, we assume that there are image currents located at the center of the cylinder, and at a distance $\frac{R}{d}$ from the center of the circle. Figure C.1 illustrates the image currents for the field outside. With these currents, the fields outside of the topological insulator are given by

$$B_r = -\frac{\mu_0 I'}{2\pi} \frac{b \sin(\theta)}{r^2 + b^2 - 2br \cos(\theta)} - \frac{\mu_0 I}{2\pi} \frac{d \sin(\theta)}{r^2 + d^2 - 2dr \cos(\theta)}$$  \hspace{1cm} (C.1)

$$B_\theta = \frac{\mu_0 I_c}{2\pi} \frac{1}{r} + \frac{\mu_0 I'}{2\pi} \frac{(r - b \cos(\theta))}{r^2 + b^2 - 2br \cos(\theta)} + \frac{\mu_0 I}{2\pi} \frac{(r - d \cos(\theta))}{r^2 + d^2 - 2dr \cos(\theta)}$$  \hspace{1cm} (C.2)
\[ E_r = \frac{\mu_0 J'}{2\pi} \frac{b \sin(\theta)}{(r^2 + b^2 - 2br \cos(\theta))} \]  
(C..3)

\[ E_\theta = -\frac{\mu_0 J_c}{2\pi} \frac{1}{r} - \frac{\mu_0 J'}{2\pi} \frac{(r - b \cos(\theta))}{(r^2 + b^2 - 2br \cos(\theta))} \]  
(C..4)

where \( b = \frac{R}{d} \). Similarly, the fields outside are determined by assuming currents lie a distance \( d \) from the edge of the cylinder. The fields take the form

\[ B_r = -\frac{\mu_0 I''}{2\pi} \frac{d \sin(\theta)}{(r^2 + d^2 - 2dr \cos(\theta))} \]  
(C..5)

\[ B_\theta = \frac{\mu_0 I''}{2\pi} \frac{(r - d \cos(\theta))}{(r^2 + d^2 - 2d \cos(\theta))} \]  
(C..6)

\[ E_r = \frac{\mu_0 J''}{2\pi} \frac{d \sin(\theta)}{(r^2 + d^2 - 2d \cos(\theta))} \]  
(C..7)

\[ E_\theta = -\frac{\mu_0 J''}{2\pi} \frac{(r - d \cos(\theta))}{(r^2 + d^2 - 2d \cos(\theta))}. \]  
(C..8)

Now first note that

\[ \frac{b \sin(\theta)}{(r^2 + b^2 - 2br \cos(\theta))} \bigg|_{r=R} = \frac{d \sin(\theta)}{(d^2 + R^2 - 2dR \cos(\theta))} \]  
(C..9)

and

\[ \frac{(r - b \cos(\theta))}{(r^2 + b^2 - 2br \cos(\theta))} \bigg|_{r=R} = \frac{1}{R} - \frac{(R - d \cos(\theta))}{(R^2 + d^2 - 2dR \cos(\theta))} \]  
(C..10)

so that at the boundary \( r = R \), the fields outside simplify to

\[ B_r = -\frac{\mu_0 I'}{2\pi} \frac{d \sin(\theta)}{(R^2 + (d)^2 - 2dR \cos(\theta))} - \frac{\mu_0 I}{2\pi} \frac{d \sin(\theta)}{(R^2 + d^2 - 2dR \cos(\theta))} \]  
(C..11)

\[ B_\theta = \frac{\mu_0 I_c}{2\pi} \frac{1}{R} + \frac{\mu_0 I'}{2\pi} \frac{1}{R} - \frac{\mu_0 I'}{2\pi} \frac{(R - d \cos(\theta))}{(R^2 + (d)^2 - 2dR \cos(\theta))} + \frac{\mu_0 I}{2\pi} \frac{(R - d \cos(\theta))}{(R^2 + d^2 - 2dR \cos(\theta))} \]  
(C..12)

\[ E_r = \frac{\mu_0 J'}{2\pi} \frac{d \sin(\theta)}{(R^2 + d^2 - 2dR \cos(\theta))} \]  
(C..13)

\[ E_\theta = -\frac{\mu_0 J_c}{2\pi} \frac{1}{R} - \frac{\mu_0 J'}{2\pi} \frac{1}{R} + \frac{\mu_0 J'}{2\pi} \frac{(R - d \cos(\theta))}{(R^2 + d^2 - 2dR \cos(\theta))}. \]  
(C..14)
The boundary conditions then become

\[ E_{\theta} \text{continuous} \]  \hspace{1cm} (C..15)

\[ B_r \text{continuous} \]  \hspace{1cm} (C..16)

\[ E_r^{\text{in}} - E_r^{\text{out}} = -c \theta_0 B_r \big|_{r=R} \]  \hspace{1cm} (C..17)

\[ B_\theta^{\text{in}} - B_\theta^{\text{out}} = \frac{\theta_0}{c} E_\theta \big|_{r=R} \]  \hspace{1cm} (C..18)

which after plugging in our values of the fields, grouping like terms, and rearranging yields

\[ -\frac{(J_c + J')}{R} + \frac{(J' + J'')(R - d \cos(\theta))}{(R^2 + d^2 - 2dR \cos(\theta))} = 0 \]  \hspace{1cm} (C..19)

\[ \frac{(I' + I - I'')(R - d \cos(\theta))}{(R^2 + d^2 - 2dR \cos(\theta))} = 0 \]  \hspace{1cm} (C..20)

\[ \frac{(J'' - J' - c \theta_0 I'')(R - d \cos(\theta))}{(R^2 + d^2 - 2dR \cos(\theta))} = 0 \]  \hspace{1cm} (C..21)

\[ \frac{(I'' + I' - I + \frac{\theta_0 J''}{c})(R - d \cos(\theta)) - (I_c + I')}{R} = 0. \]  \hspace{1cm} (C..22)

We require this to be true for all \( \theta \), so that the coefficients of each angular term must be zero separately. This yields the following system of equations:

\[ J_c + J' = 0 \]  \hspace{1cm} (C..23)

\[ J' + J'' = 0 \]  \hspace{1cm} (C..24)

\[ I' + I - I'' = 0 \]  \hspace{1cm} (C..25)

\[ J'' - J' - c \theta_0 I'' = 0 \]  \hspace{1cm} (C..26)

\[ I'' + I' - I + \frac{\theta_0}{c} J'' = 0 \]  \hspace{1cm} (C..27)

\[ I_c + I' = 0. \]  \hspace{1cm} (C..28)
Solving these equations gives the following values for the currents:

\[ I' = -\frac{\theta_0^2}{4} \frac{I}{1 + \frac{\theta_0^2}{4}} \]  
(C..29)

\[ I'' = \frac{1}{1 + \frac{\theta_0^2}{4}} I \]  
(C..30)

\[ I_c = \frac{\theta_0^2}{4} \frac{I}{1 + \frac{\theta_0^2}{4}} \]  
(C..31)

\[ J' = -\frac{\theta_{ac}}{2} \frac{I}{1 + \frac{\theta_0^2}{4}} \]  
(C..32)

\[ J'' = \frac{\theta_{ac}}{2} \frac{I}{1 + \frac{\theta_0^2}{4}} \]  
(C..33)

\[ J_c = \frac{\theta_{ac}}{2} \frac{I}{1 + \frac{\theta_0^2}{4}}. \]  
(C..34)
D. Spherical Topological Insulator in Constant Electric and Magnetic Fields

In this section, we solve the problem of a topological insulator in a constant electric and magnetic field. First we handle each case separately, then we move on to the case in which the fields point in the same direction.

![Diagram of a sphere in a constant electric field](image)

Figure D..1: Spherical Topological Insulator in a Constant E field

To begin, consider the problem of a sphere of radius $a$ in a constant electric field. We choose coordinate axes such that the field points along the z-direction and the origin coincides with the center of the sphere. Figure D..1 illustrates the problem being considered.

The spherical and azimuthal symmetry of the problem allows for an expansion in Legendre Polynomials. In spherical coordinates, the potential associated with the applied field is given by

$$\Phi_{app} = -E_0 z = -E_0 r \cos(\theta) = -E_0 r P_1(\cos(\theta)).$$  \hspace{1cm} (D..1)

The fact that the applied field only contributes an $\ell = 1$ term implies that all other terms drop out of the expansion. Therefore, we need only consider an expansion in the first legendre polynomial.
In addition, we require the solution to be regular at the origin, and to reduce to the constant field as \( r \to \infty \). Putting all of this together, the potentials inside and outside of the topological Insulator take the form

\[
\Phi^\text{out}_E = \left( \frac{A}{r^2} - E_0 r \right) P_1(\cos(\theta)) \\
\Phi^\text{out}_B = \frac{B}{r^2} P_1(\cos(\theta)) \\
\Phi^\text{in}_E = C r P_1(\cos(\theta)) \\
\Phi^\text{in}_B = D r P_1(\cos(\theta))
\]

The boundary conditions then become

\[
\frac{\partial \Phi^\text{in}_E}{\partial \theta} = \frac{\partial \Phi^\text{out}_E}{\partial \theta} \bigg|_{r=a} \quad (D.6)
\]
\[
\frac{\partial \Phi^\text{in}_B}{\partial r} = \frac{\partial \Phi^\text{out}_B}{\partial r} \bigg|_{r=a} \quad (D.7)
\]
\[
- \frac{\partial \Phi^\text{in}_E}{\partial r} + \frac{\partial \Phi^\text{out}_E}{\partial r} = \theta_0 c \frac{\partial \Phi^\text{in}_B}{\partial r} \bigg|_{r=a} \quad (D.8)
\]
\[
- \frac{\partial \Phi^\text{in}_B}{\partial \theta} + \frac{\partial \Phi^\text{out}_B}{\partial \theta} = - \frac{\theta_0}{c} \frac{\partial \Phi^\text{in}_E}{\partial \theta} \bigg|_{r=a} \quad (D.9)
\]

Solving this system of equations yields the following values for the unknown constants \( A, B, C, D \):

\[
A = \frac{2 \theta_0^2}{(2 \theta_0^2 + 9)} E_0 a^3 \quad (D.10)
\]
\[
B = - \frac{3 \theta_0}{c(2 \theta_0^2 + 9)} E_0 a^3 \quad (D.11)
\]
\[
C = - \frac{9}{(2 \theta_0^2 + 9)} E_0 \quad (D.12)
\]
\[
D = - \frac{6 \theta_0}{c(2 \theta_0^2 + 9)} E_0 \quad (D.13)
\]
so that the potentials take the form

\[ \Phi_{\text{out}}^E = \left( \frac{2\theta^2_0}{(2\theta^2_0 + 9)} r^2 - r \right) E_0 P_1(\cos(\theta)) \] (D..14)

\[ \Phi_{\text{out}}^B = -\frac{3\theta^2_0}{c(2\theta^2_0 + 9)} a^3 E_0 P_1(\cos(\theta)) \] (D..15)

\[ \Phi_{\text{in}}^E = -\frac{9}{(2\theta^2_0 + 9)} E_0 r P_1(\cos(\theta)) \] (D..16)

\[ \Phi_{\text{in}}^B = -\frac{6\theta_0}{c(2\theta^2_0 + 9)} B_0 r P_1(\cos(\theta)). \] (D..17)

Indeed the form of the electric potential very much resembles the potentials that occur when a dielectric sphere is immersed in a constant electric field. What is interesting to note is that in the case of a topological insulator, a magnetic dipole like term forms.

If we consider the same problem with a constant magnetic field rather than a constant electric field, a very similar reaction occurs. The procedure is identical to that of the constant electric field and will not be shown. The final form of the potentials is

\[ \Phi_{\text{out}}^E = \frac{3\theta_0 c}{(2\theta^2_0 + 9)} a^3 B_0 P_1(\cos(\theta)) \] (D..18)

\[ \Phi_{\text{out}}^B = \left( -\frac{\theta^2_0}{(2\theta^2_0 + 9)} a^3 \right) r B_0 P_1(\cos(\theta)) \] (D..19)

\[ \Phi_{\text{in}}^E = \frac{3\theta_0 c}{(2\theta^2_0 + 9)} B_0 r P_1(\cos(\theta)) \] (D..20)

\[ \Phi_{\text{in}}^B = -\frac{9}{(2\theta^2_0 + 9)} B_0 r P_1(\cos(\theta)). \] (D..21)

An interesting thing to note about the sphere in a constant magnetic field is the form of the electric potential inside. The positive nature of the electric potential inside the sphere indicates that the electric field is negative inside the sphere. This sort of response is analogous to the dielectric sphere in a constant electric field if we allow the dielectric constant of the sphere to be negative, something not normally found in nature.

We can also consider the case in which a spherical topological insulator is immersed in a
constant electric and magnetic field that are in the same direction. The result is a superposition of the separate cases. The potentials are listed below:

\[
\Phi_{\text{out}}^{E} = \left( \frac{2\theta_0^2}{(2\theta_0^2 + 9)} \frac{a^3}{r^2} E_0 + \frac{3\theta_0 c}{(2\theta_0^2 + 9)} \frac{a^3}{r} B_0 - E_0 r \right) P_1(\cos(\theta)) \tag{D.22}
\]

\[
\Phi_{\text{out}}^{B} = -\left( \frac{3\theta_0}{c(2\theta_0 + 9)} \frac{a^3}{r^2} E_0 + \frac{\theta_0^2}{(2\theta_0^2 + 9)} \frac{a^3}{r^2} B_0 + B_0 r \right) P_1(\cos(\theta)) \tag{D.23}
\]

\[
\Phi_{\text{in}}^{E} = \left( \frac{3\theta_0 c}{(2\theta_0^2 + 9)} B_0 - \frac{9}{(2\theta_0^2 + 9)} E_0 \right) r P_1(\cos(\theta)) \tag{D.24}
\]

\[
\Phi_{\text{in}}^{B} = -\left( \frac{6\theta_0}{c(2\theta_0^2 + 9)} E_0 + \frac{9}{(2\theta_0^2 + 9)} B_0 \right) r P_1(\cos(\theta)). \tag{D.25}
\]