
On the simulation of expanding spacetimes in trapped Bose-Einstein condensates

Michael Uhlmann

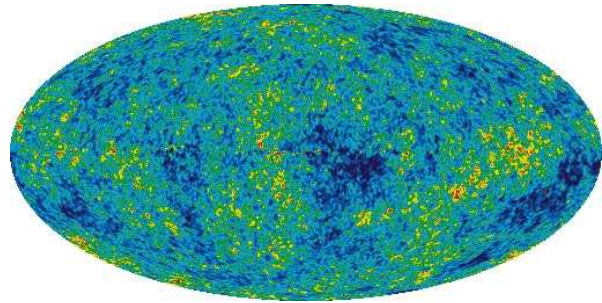
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Cleveland, Ohio

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Motivation

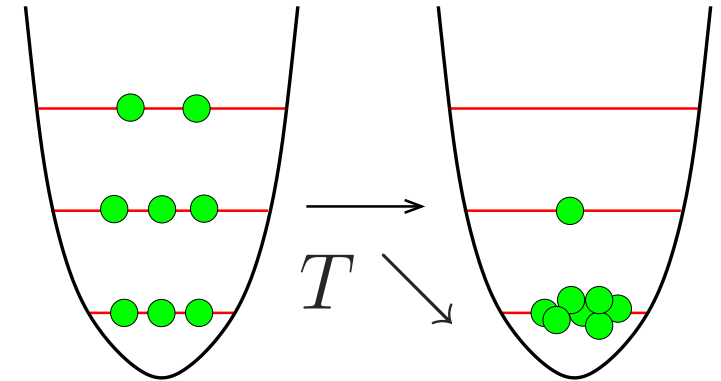


Low-energy phase fluctuations in fluid obey same evolution equations as **massless scalar field** in a certain curved spacetime

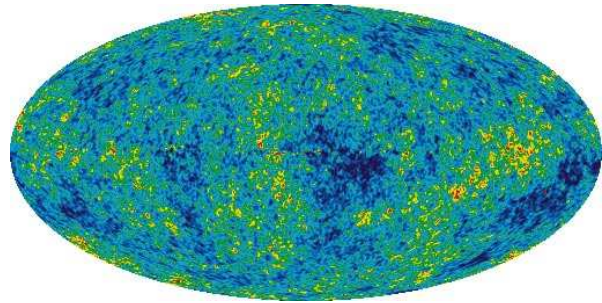
Universal features of quantum effects
(e.g., adiabatic theorem)

Bose-Einstein condensates

$$i\partial_t \hat{\Psi} = \left[-\frac{\nabla^2}{2} + V_{\text{ext}} + g\hat{\Psi}^\dagger \hat{\Psi} \right] \hat{\Psi}$$



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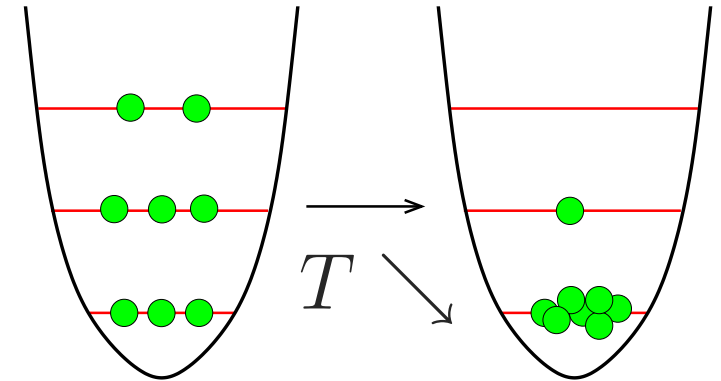
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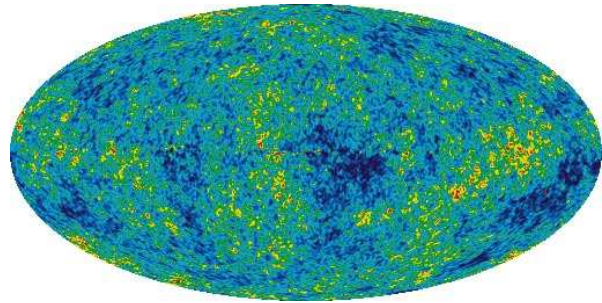
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Change interaction strength $g(t)$

Vary external trap potential $V_{\text{ext}}(\mathbf{r}, t)$



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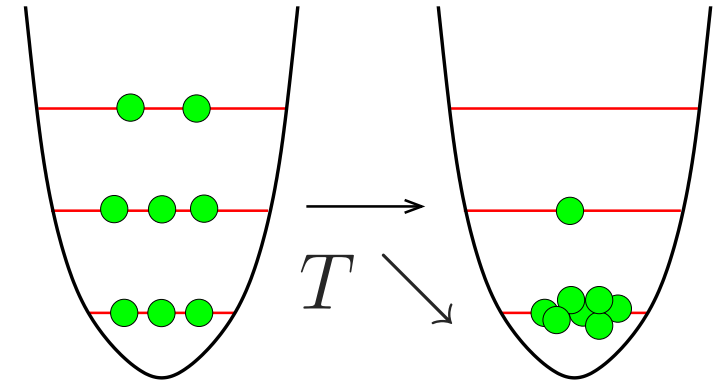
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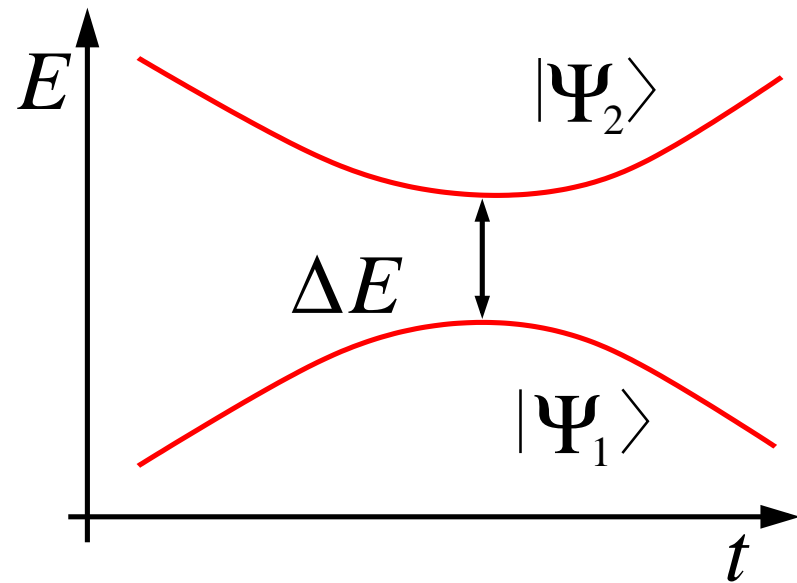
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Simple experiment: Tune $g(t)$ over huge range
Measure density-density correlations

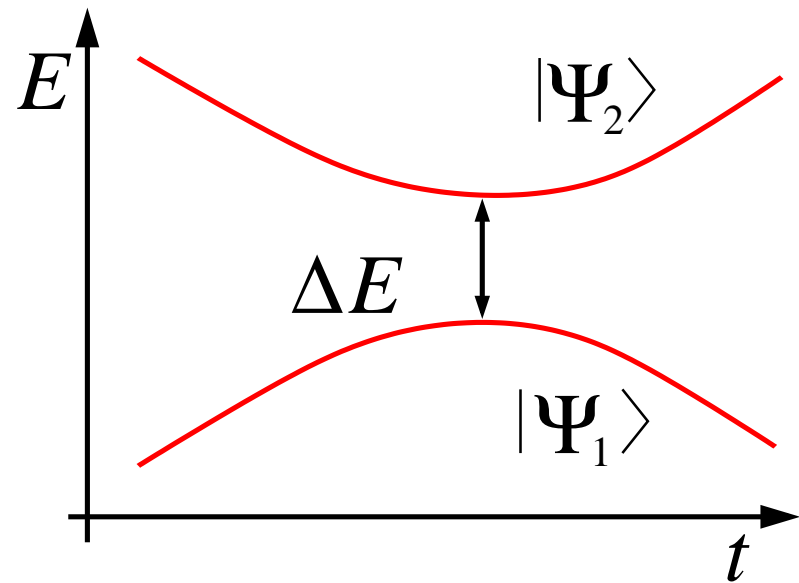
Adiabatic Theorem



2 states $|\Psi_1\rangle$ and $|\Psi_2\rangle$
with time-dependent gap $\Delta E(t)$

Adiabatic if $\Delta E \gg 1/t_{\text{ext}}$
Else: transitions, excitations

Adiabatic Theorem



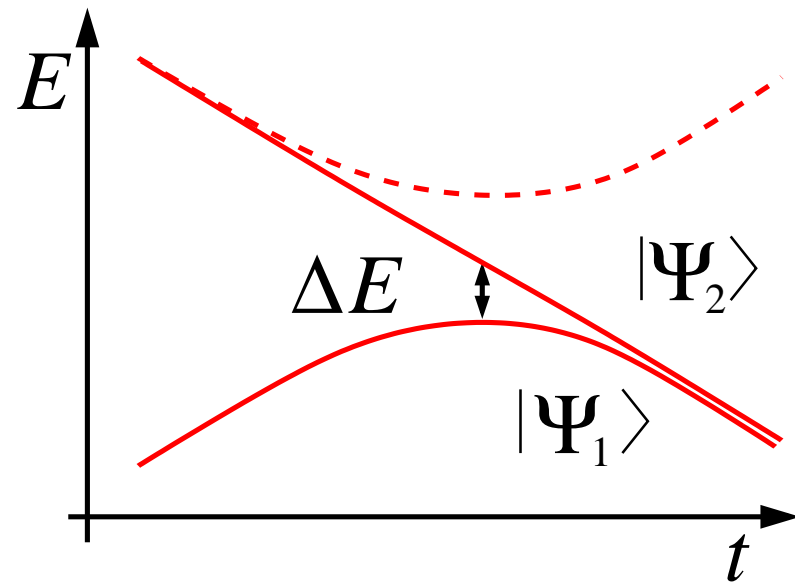
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Scalar field in expanding spacetime, e.g., de Sitter $a = e^{Ht}$

- Physical wavelength grows $\lambda_{\text{phys}} = a(t)\lambda_{\text{comoving}}$
- Nonadiabatic evolution when $\lambda_{\text{phys}} \gtrsim 1/H$

$$\left[\frac{\partial^2}{\partial t^2} + 3H \frac{\partial}{\partial t} - \frac{\nabla^2}{a^2} \right] \hat{\phi} = 0$$

Effective Spacetime

Split field operator $\hat{\Psi} = \Psi_0 + \delta\hat{\Psi}$ into modulus and phase

$$\Psi_0 = e^{i\Phi_0} \sqrt{\varrho_0}, \quad \delta\hat{\psi} = \Psi_0 \left(\frac{\delta\hat{\varrho}}{2\varrho_0} + i\delta\hat{\Phi} \right)$$

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Bernoulli and continuity equations for background $\mathbf{v}_0 = \nabla\Phi_0$

$$\text{Equation of state } \nabla p = \varrho_0 \nabla [V_{\text{ext}} - (\nabla^2 \sqrt{\varrho_0}) / 2\sqrt{\varrho_0} + g\varrho_0]$$

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If $[\nabla^2 \sqrt{\varrho_0}]/\sqrt{\varrho_0}$ negligible: **effective space-time**

$$\partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu \delta\hat{\Phi} = 0, \quad g_{\mu\nu} = \frac{1}{A_D c_s^2} \begin{pmatrix} c_s^2 - \mathbf{v}_0^2 & v_0^i \\ v_0^j & -\delta_{ij} \end{pmatrix}$$

with sound velocity

$$c_s^2 = g\varrho_0,$$

and prefactor

$$A_D = (c_s/g)^{2/(D-1)}$$

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Simulate aspects of cosmic quantum effects

Apply concepts from general relativity

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In experiments usually external confinement
i.e., discrete excitation spectrum

Background motion?

Background motion

Field equation for background with harmonic trap

$$i\partial_t\Psi_0 = \left[-\frac{\nabla^2}{2} + \frac{\omega^2(t)r^2}{2} + g(t)|\Psi_0|^2 \right] \Psi_0$$

Transformation to (almost) **comoving coordinates**

$$\mathbf{x} = \mathbf{r}/b(t) \text{ and } \psi_0 = e^{i\Phi}\Psi_0/b^{D/2}$$

$$ib^2(t)\partial_t\psi_0 = \left[-\frac{\nabla_x^2}{2} + f^2(t) \left(\frac{\mathbf{x}^2}{2} + g_0|\psi_0|^2 \right) \right] \psi_0$$

$$\text{with } f^2(t) = b^3 \frac{\partial^2}{\partial t^2} b + b^4 \omega^2(t) = \frac{g(t)}{g_0} b^{2-D}$$

Most of background motion contained in $f^2(t)$
small density ripples near surface of condensate

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Note $[\nabla_x^2 \sqrt{\varrho_0}] / \sqrt{\varrho_0}$ needs to be small for effective spacetime

Quantum fluctuations

Self-adjoint operators

$$\hat{\chi}_+ = e^{-i\Phi_0} \delta\hat{\psi} + e^{i\Phi_0} \delta\hat{\psi}^\dagger = \frac{\delta\hat{\varrho}}{\sqrt{\varrho_0}}$$

$$\hat{\chi}_- = \frac{1}{2i} \left(e^{-i\Phi_0} \delta\hat{\psi} - e^{i\Phi_0} \delta\hat{\psi}^\dagger \right) = \sqrt{\varrho_0} \delta\hat{\phi}$$

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Initial eigenmode expansion $\hat{\chi}_\pm = h_n^\pm(\mathbf{x}; t_0) \hat{X}_n^\pm(t)$

with duality $\int h_n^+(\mathbf{x}) h_m^-(\mathbf{x}) = \delta_{nm}$

Evolution equations

$$b^2 \frac{\partial}{\partial t} \hat{X}_n^- = -\frac{1}{2} \mathcal{A}_{nm}(t) \hat{X}_m^+ - \mathcal{V}_{nm}(t) \hat{X}_m^-$$

$$b^2 \frac{\partial}{\partial t} \hat{X}_n^+ = 2\mathcal{B}_{nm}(t) \hat{X}_m^- + \mathcal{V}_{nm}(t) \hat{X}_m^+$$

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Evolution equations are diagonal at t_0 (when $\mathcal{V}_{nm} = 0$)

$$b^2 \frac{\partial}{\partial t} \hat{X}_n^- = -\frac{1}{2} \mathcal{A}_{nm}(t) \hat{X}_m^+ - \mathcal{V}_{nm}(t) \hat{X}_m^- \stackrel{t=t_0}{=} -\frac{1}{2} A_n \hat{X}_n^+$$

$$b^2 \frac{\partial}{\partial t} \hat{X}_n^+ = 2\mathcal{B}_{nm}(t) \hat{X}_m^- + \mathcal{V}_{nm}(t) \hat{X}_m^+ \stackrel{t=t_0}{=} 2B_n \hat{X}_n^-$$

Coupling matrices

$$\mathcal{A}_{nm} = \int h_n^+ \left(-\frac{\nabla_x^2}{2} + 2f^2 g_0 \varrho_0 + \frac{\nabla_x^2 \sqrt{\varrho_0}}{2\sqrt{\varrho_0}} \right) h_m^+$$

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Decoupling of modes n and m if

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Decoupling of modes n and m if

First integral dominated by $f^2 g_0 \varrho_0$

Time-dependence of ϱ_0 must separate in support of modes
 \Rightarrow Centre of the trap and low energies

Background motion described by $f^2(t)$

Thomas-Fermi approximation

Evolution equation for low modes ($\hat{\chi}_- = \sqrt{\varrho_0} \delta \hat{\phi}$, $d\tau = dt/b^2$)

$$\left[\frac{\partial^2}{\partial \tau^2} - 2 \frac{\partial \ln f}{\partial \tau} \frac{\partial}{\partial \tau} + f^2 A_n B_n \right] \delta \hat{\phi}_n = 0$$

Same as for a (massless) **scalar field mode** in a (flat) **Friedmann-Robertson-Walker-Lemaître spacetime**

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(Static) Thomas-Fermi profile

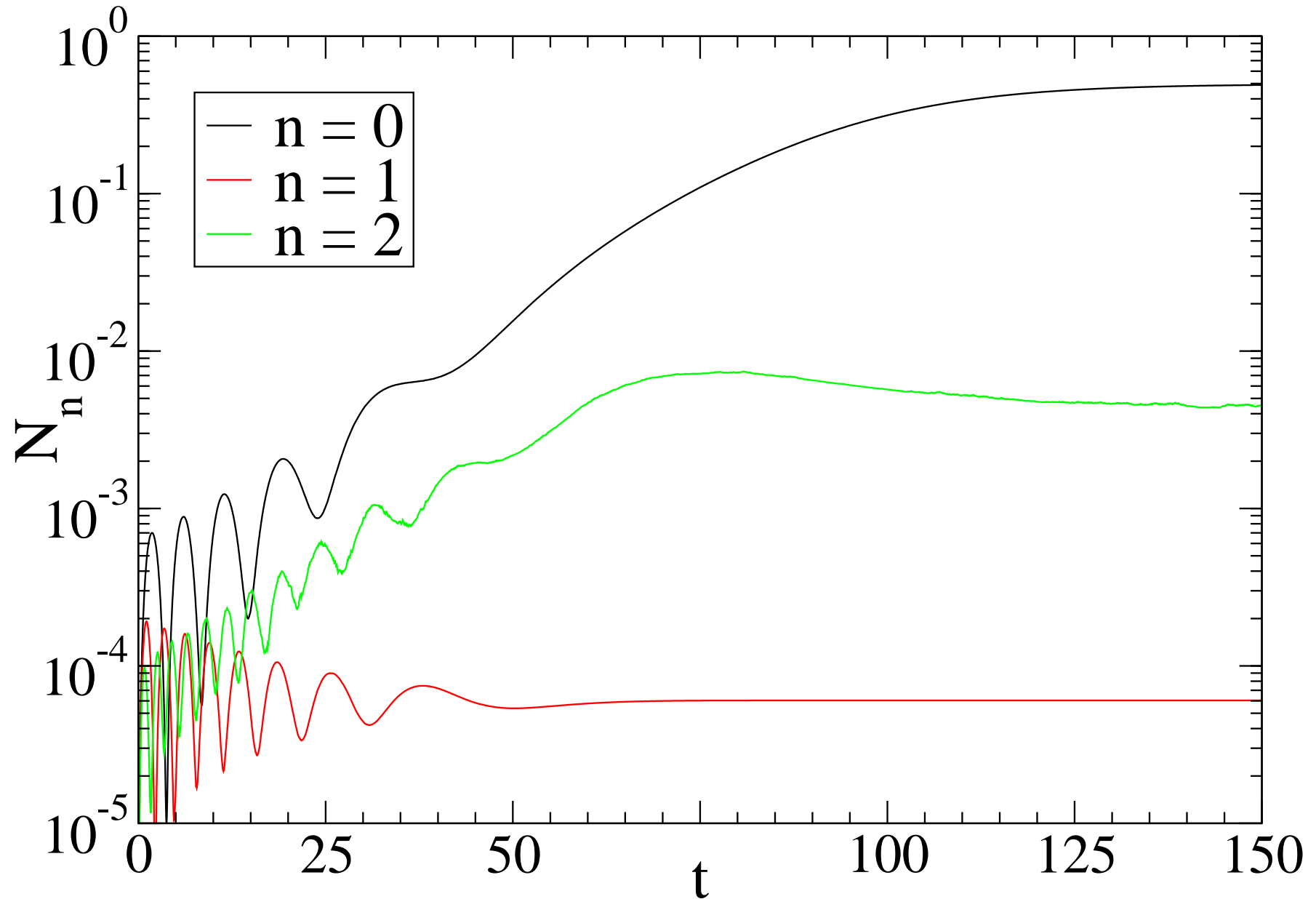
$$\mu_0 = \frac{\omega^2 r^2}{2} + g \varrho_0 ,$$

Sound velocity

$$c_s^2 = g \varrho_0$$

Comparison $r_{\text{TF}}^2 = \frac{2}{\omega^2} c_s^2 (r = 0)$

Exponential sweep, $g = \omega = e^{-2\gamma t}$, $\gamma = 0.1$



Summary and conclusions

Universal features of quantum effects

Adiabatic theorem

Quantum fluctuations in trapped
Bose-Einstein condensate



Australian Government

Australian Research Council

**Effective spacetime for low-lying
excitations $\Omega_n \ll \mu_0$**

Intermode coupling for $\Omega_n \gtrsim \mathcal{O}(\mu_0)$

For realistic variations $g(t)$:

**Need to change trap frequency $\omega(t)$ as
well or adiabaticity not violated**



Alexander von Humboldt
Stiftung/Foundation