Spatially Covariant Theories of a Transverse, Traceless Graviton

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Why modify gravity?
Cosmic Acceleration

Saul Perlmutter, Brian P. Schmidt, Adam G. Riess

“for the discovery of the accelerating expansion of the Universe through observations of distant supernovae”

\[ \ddot{a} > 0 \]

2011
Implications of Acceleration

\[ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3\rho) \]

\[ \ddot{a} > 0 \quad \implies \quad p < -\frac{1}{3} \rho \]

Cosmological Constant

\[ S = \int d^4x \sqrt{-g} \Lambda \quad \implies \quad \rho\Lambda = -\rho\Lambda \]

\( \Lambda \text{CDM}, \text{the Concordance Model of Cosmology} \)

\[ \Omega_{\Lambda} \approx 0.7 \quad \implies \quad \rho_{\Lambda} \approx (meV)^4 \]
Cosmological Constant Problems

**Expectation**
\[ \rho_\Lambda \approx M_{Pl}^4 \]

**Apparent Reality**
\[ \rho_\Lambda \approx (meV)^4 \approx 10^{-120} M_{Pl}^4 \]

**Cosmological Constant Problem** – Why is \( \rho_\Lambda \) so small?

- Cancellation of zero-point energies requires extreme fine-tuning
- Weinberg’s “No-Go” Theorem: Cannot relax dynamically to \( \rho_\Lambda \approx 0 \)

S. Weinberg, Rev.Mod.Phys. 61 (1989) 1-23

**Coincidence Problem** – Why is the dark energy density comparable to the present matter density?

- Weinberg’s second “No-Go” Theorem: Cannot relax to \( \rho_\Lambda \approx \rho_m \) today without extreme fine-tuning

S. Weinberg, astro-ph 0005265
Common Modifications

- Scalar Tensor, $f(R)$, higher order invariants
- Massive gravity
- Braneworld scenarios: DGP, Cascading gravity
- Ghost condensation
- Galileons, Chameleons, Symmetrons
- and many more!
Common Modifications

- Scalar Tensor, f(R), higher order invariants
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- and many more!

Each of these proposals adds new degrees of freedom to GR
Degrees of Freedom

Field theories have infinite degrees of freedom, but...

Finitely many local degrees of freedom, which count particle polarization states

In Lorentz covariant field theories, the polarization states are fixed by the mass and spin of the particle excitations

A real scalar field has one, a massless vector field has two, a massive vector field has three, etc.
Gravitons

General Relativity has two degrees of freedom: the polarizations of the transverse, traceless graviton, a massless spin-2 particle.

Many modifications of GR introduce new particles: Chamelons, Symmetrons, Galileons, etc.

Some modifications introduce new graviton polarizations: for example, a massive graviton would have five polarizations.
Gravitons

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Could General Relativity be modified without new degrees of freedom?
Can we modify the behavior of the transverse, traceless graviton?
Weinberg's Theorem: GR is the unique Lorentz covariant theory of an interacting massless spin-2 particle.
Uniqueness Theorem

Weinberg’s Theorem: GR is the unique Lorentz covariant theory of an interacting massless spin-2 particle

S. Weinberg
Phys. Rev. 138 (1965) B988-B1002

S. Deser
gr-qc/0411023

Lorentz covariant modifications of GR introduce new degrees of freedom
Uniqueness Theorem

Weinberg's Theorem: GR is the unique Lorentz covariant theory of an interacting massless spin-2 particle.

Lorentz covariant modifications of GR introduce new degrees of freedom.

In particular, theories that break Lorentz symmetry spontaneously (e.g., ghost condensation) always introduce new degrees of freedom.

The additional degrees of freedom manifest as massless Goldstone bosons in the broken phase.

S. Weinberg
Phys. Rev. 138 (1965) B988-B1002

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gr-qc/0411023
Modifying the Graviton

Binary pulsars offer strong evidence for the two graviton polarizations of GR.

Lorentz covariant modifications of GR introduce new degrees of freedom, but we haven’t seen any.

To modify the behavior of the graviton without new degrees of freedom, what must we do?
Modifying the Graviton

- Binary pulsars offer strong evidence for the two graviton polarizations of GR

- Lorentz covariant modifications of GR introduce new degrees of freedom, but we haven’t seen any

- To modify the behavior of the graviton without new degrees of freedom, what must we do?

  Break Lorentz covariance...
Modifying the Graviton

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Lorentz covariant modifications of GR introduce new degrees of freedom, but we haven’t seen any.

To modify the behavior of the graviton without new degrees of freedom, what must we do?

Break Lorentz covariance... Explicitly!
Breaking Covariance

Lorentz covariance is a very well-motivated assumption, but...
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Is the gravitational sector Lorentz covariant?
Breaking Covariance

Lorentz covariance is a very well-motivated assumption, but...

Is the gravitational sector Lorentz covariant?

Hard to experimentally verify the Lorentz covariance of the graviton S-matrix - same problem arises in neutrino physics.

Given the enduring mystery of dark energy, we may need to revisit the assumption of spacetime symmetry.

Might be useful to know what deviations from Lorentz covariance would look like cosmologically.
Breaking Covariance

- Lorentz covariance is a very well-motivated assumption, but...

  **Is the gravitational sector Lorentz covariant?**

- Hard to experimentally verify the Lorentz covariance of the graviton S-matrix - same problem arises in neutrino physics

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- Might be useful to know what deviations from Lorentz covariance would look like cosmologically

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After all, our universe has a preferred frame
Cosmic Rest Frame

"for their discovery of the blackbody form and anisotropy of the cosmic microwave background radiation"

Spontaneous or Explicit Symmetry Breaking?
Careful Breaking

- We will break explicit Lorentz symmetry
- We will preserve explicit spatial symmetry
Careful Breaking

- We will **break** explicit Lorentz symmetry
- We will preserve explicit **spatial** symmetry

**Caution:** Breaking a symmetry can introduce new degrees of freedom, or render the theory **inconsistent**
Careful Breaking

- We will break explicit Lorentz symmetry
- We will preserve explicit spatial symmetry

**Caution:** Breaking a symmetry can introduce new degrees of freedom, or render the theory inconsistent

- To analyze symmetries and count degrees of freedom, we will use constrained field theory
- To avoid new degrees of freedom, we must preserve the balance between the size of phase space and the number of constraints
In canonical form, GR is a theory of a spatial metric $\tilde{h}_{ij}$ (which has six components) subject to four spacetime gauge symmetries:

$$6 - 4 = 2$$

For theories of a spatial metric, relaxing four spacetime symmetries to three spatial symmetries introduces a new graviton polarization:

$$6 - 3 = 3$$

e.g., P. Horava

Instead, consider theories of a unit-determinant spatial metric $\tilde{h}_{ij}$ (which has five components) subject to three spatial gauge symmetries:

$$5 - 3 = 2$$

In a particular gauge, general relativity can be cast in this form!
ADM Action for GR

**Einstein-Hilbert Action**

\[ S = \int d^4x \sqrt{-g} \left( R^{(4)} - 2\Lambda \right) \]

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \]

\[ = -N^2 dt^2 + h_{ij} (dx^i + N^i dt)(dx^j + N^j dt) \]

**Arnowitt-Deser-Misner Decomposition**

\[ S = \int dt d^3x N \sqrt{h} \left( K_{ij} K_{ij} - K^2 + R^{(3)} - 2\Lambda \right) \]

**Extrinsic Curvature**

\[ K_{ij} \equiv \frac{1}{2} N^{-1} \left( \dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i \right) \]

Only \( h_{ij} \) is dynamical
Canonical Action for GR

\[ \pi^{ij} \equiv \frac{\delta L}{\delta h_{ij}} = \sqrt{h} \left( K^{ij} - h^{ij} K \right) \]

\[ S = \int dt \, d^3x \left( \dot{h}_{ij} \pi^{ij} - N^\mu \mathcal{H}_\mu \right) \]

\[ N^0 \equiv N \]

The \( N^\mu \)’s are Lagrange multipliers enforcing \( \mathcal{H}_\mu \sim 0 \)

The graviton is:

**Traceless**

\[ \mathcal{H}_0 \equiv \frac{1}{\sqrt{h}} \left( \pi^{ij} \pi_{ij} - \frac{1}{2} (\pi^i_i)^2 \right) + \sqrt{h} \left( 2\Lambda - R^{(3)} \right) \]

**Transverse**

\[ \mathcal{H}_i \equiv -2 \dot{h}_{ij} \nabla^k \pi^{jk} \]

Poisson Bracket

\[ \{ A, B \} \equiv \int d^3z \left( \frac{\delta A}{\delta h_{mn}(z)} \frac{\delta B}{\delta \pi^{mn}(z)} - \frac{\delta A}{\delta \pi^{mn}(z)} \frac{\delta B}{\delta h_{mn}(z)} \right) \]
Constrained Field Theory

\[ S = \int dt \, d^3 x \left( \dot{h}_{ij} \pi^{ij} - N^\mu \mathcal{H}_\mu \right) \]

- Lagrange Multipliers
- Constraints

- Constraints \( \mathcal{H}_\mu \) define a surface in phase space, the constraint surface.

- We introduce the symbol \( \sim \) to denote "equality on the constraint surface" or weak equality.

\[ H = \int d^3 x \, N^\mu \mathcal{H}_\mu \sim 0 \]

Physical Hamiltonian
Constraint Classes

- Constraints $\psi_a$ split into two classes
  - **First class** constraints $U_A$
  - **Second class** constraints $V_M$

\[
\{U_A, U_B\} = f_{AB}^C U_C \sim 0
\]
\[
\{U_A, V_M\} = \theta_{AM}^C \psi_C \sim 0
\]
\[
\{V_M, V_N\} = C_{MN} \sim 0
\]

- First class constraints generate **gauge symmetries**
- Second class constraints are **gauge-fixed** constraints
- As we will see, GR contains only first class constraints - in other words, GR is a **gauge theory**

P.A.M. Dirac
GR Constraint Algebra

After much labor, one can show that

\[
\{ \mathcal{H}_0(x), \mathcal{H}_0(y) \} = \mathcal{H}^i(x) \partial_{x^i} \delta^3(x - y) - \mathcal{H}^i(y) \partial_{y^i} \delta^3(x - y)
\]

\[
\{ \mathcal{H}_0(x), \mathcal{H}_1(y) \} = \mathcal{H}_0(y) \partial_{x^i} \delta^3(x - y)
\]

\[
\{ \mathcal{H}_1(x), \mathcal{H}_j(y) \} = \mathcal{H}_j(x) \partial_{x^i} \delta^3(x - y) - \mathcal{H}_j(y) \partial_{y^i} \delta^3(x - y)
\]

This algebra is **first class**, i.e.,

\[
\mathcal{H}_\mu \sim 0 \quad \rightarrow \quad \{ \mathcal{H}_\mu(x), \mathcal{H}_\nu(y) \} \sim 0
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GR Constraint Algebra

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\{\mathcal{H}_0(x), \mathcal{H}_1(y)\} = \mathcal{H}_0(y) \partial_{x^i} \delta^3(x - y)
\]

\[
\{\mathcal{H}_1(x), \mathcal{H}_j(y)\} = \mathcal{H}_j(x) \partial_{x^i} \delta^3(x - y) - \mathcal{H}_1(y) \partial_{y^j} \delta^3(x - y)
\]

This algebra is \textbf{first class}, i.e.,

\[
\mathcal{H}_\mu \sim 0 \quad \Rightarrow \quad \{\mathcal{H}_\mu(x), \mathcal{H}_\nu(y)\} \sim 0
\]

What is the gauge symmetry?
The $\mathcal{H}_\mu$'s generate the deformations of a spacelike hypersurface in a Riemannian spacetime.

This "general covariance" algebra encodes the local Lorentz covariance of a canonical action.

GR is the unique minimal representation; this result complements Weinberg's Theorem.
Consistency of constraints with equations of motion requires $\dot{H}_\mu \sim 0$

$$H = \int d^3 x \, N^\mu \mathcal{H}_\mu \sim 0$$

$$\dot{A} = \frac{\partial A}{\partial t} + \{A, H\}$$

$$= \frac{\partial A}{\partial t} + \int d^3 y \, N^\nu (y) \{A, \mathcal{H}_\nu (y)\}$$

$$\dot{\mathcal{H}}_\mu (x) = \int d^3 y \, N^\nu (y) \{\mathcal{H}_\mu (x), \mathcal{H}_\nu (y)\}$$

$\rightarrow \dot{H}_\mu \sim 0$
Degrees of Freedom

- Phase Space: \((h_{ij}, \pi^{ij})\)
- Constraints: \(\mathcal{H}_\mu\)
- Arbitrary Functions to be gauge-fixed: \(N^\mu\)

\[6 \ h_{ij}'s + 6 \ \pi^{id}'s - 4 \ \mathcal{H}_\mu' s - 4 \ N^\mu' s = 4 \ \text{canonical degrees of freedom}\]

Two real degrees of freedom: the polarizations of the transverse, traceless graviton

To avoid new degrees of freedom, we must preserve the balance between the size of phase space and the number of constraints.
Examples

Ultralocal Limit of GR

- Neglect spatial derivatives in $H_0$
- Same phase space, number of constraints as GR

$\{H_0(x), H_0(y)\} = 0 \quad \text{Abelian Time Translation}$

$\{H_0(x), H_i(y)\} = H_0(y) \partial_{x^i} \delta^3(x - y)$

$\{H_i(x), H_j(y)\} = H_j(x) \partial_{x^i} \delta^3(x - y) - H_i(y) \partial_{y^j} \delta^3(x - y)$

Preserves Spatial Covariance

"Covariant" Horava-Lifshitz Gravity

- Larger phase space, more constraints than GR
- Obeys "non-relativistic covariance algebra" analogous to algebra of ultralocal GR

D. Salopek

P. Horava, C. M. Melby-Thompson
1007.2410
Modifying the Graviton

The Hamiltonian Constraint

- Ideally, one would like to solve the constraints $H_\mu$, modify the equations of motion on the physical phase space of the graviton.
- In general, cannot solve the Hamiltonian constraint $H_0$.
- This is an obstacle to canonical quantum gravity.
Modifying the Graviton

The Hamiltonian Constraint

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- In general, cannot solve the Hamiltonian constraint $H_0$.
- This is an obstacle to canonical quantum gravity.

Our Approach

- Solve Hamiltonian constraint in a cosmologically motivated gauge, reduce size of phase space, retain momentum constraints $H_i$.
- Modify the ensuing spatially covariant theory.
Flat FRW Metric

\[ ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \]

\[ H_{H_{ub}} \equiv \frac{\dot{a}}{a} \]

“Phase space” is \((a, H_{H_{ub}})\)

Scale factor is analogous to a canonical coordinate

Hubble parameter is analogous to a canonical momentum
Flat FRW Metric

\[ ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j \]

\[ H_{H \mu \nu} \equiv \frac{\dot{a}}{a} \]

"Phase space" is \((a, H_{H \mu \nu})\)

Scale factor is analogous to a canonical coordinate

Hubble parameter is analogous to a canonical momentum

\[ N = 1 \quad N^i = 0 \quad h_{ij} = a^2(t) \delta_{ij} \]

\[ \sqrt{h} = a^3 \quad \pi^i_i = -6a^3 \cdot H_{H \mu \nu} \]

Conformal part of spatial metric is conjugate to the trace of the momentum tensor
Conformal Decomposition

Split spatial metric $h_{ij}$ into a volume factor $\omega$ and a unit-determinant metric $\tilde{h}_{ij}$.

**Conformal Phase Space**

$\omega \equiv \sqrt{h}$

$\pi_\omega = \frac{2\pi^i_i}{3\omega}$

**Unit-Determinant Phase Space**

$\det \tilde{h}_{ij} = 1$

$h_{ij} \pi^{ij} = 0$

$h_{ij} \equiv \tilde{h}_{ij} \cdot \omega^{2/3}$

$\pi^{ij} = \frac{\tilde{\pi}^{ij}}{\omega^{2/3}} + \frac{1}{2} \tilde{h}_{ij} \cdot \pi_\omega \cdot \omega^{1/3}$

$(h_{ij}, \pi^{ij}) \quad \rightarrow \quad (\omega, \pi_\omega), \left(\tilde{h}_{ij}, \tilde{\pi}^{ij}\right)$

This decomposition is completely general.
Philosophy

- Represent time symmetry on conformal phase space, i.e., use $\omega$ as our clock

- Represent spatial symmetry on unit-determinant phase space

$$\mathcal{H}_\mu \quad (h_{ij}, \pi^{ij})$$

$$\mathcal{H}_0 \quad \omega, \pi_{\omega}$$

$$\mathcal{H}_i \quad (\tilde{h}_{ij}, \tilde{\pi}^{ij})$$

- We can pick a gauge in which the conformal phase space of GR to be non-dynamical

- We will impose this gauge with the Dirac procedure
Cosmological Gauge

- Use $\omega$ as our clock; valid about an FRW background, where $\omega$ evolves monotonically

\[ \chi \equiv \omega - \omega(t) \]

- Impose $\chi \sim 0$ with a Lagrange multiplier $\lambda$

\[ \{\chi, \mathcal{H}_0\} \sim 0 \quad \rightarrow \quad \chi, \mathcal{H}_0 \text{ are second class} \]

- The second class property allows us to solve for the corresponding Lagrange multipliers $N$ and $\lambda$

- Gauge-fixing constraint $\chi$ eliminates arbitrary function $N$, preserves counting of DOF
Phase Space Reduction

\[ H_0 = \frac{\tilde{\pi}^{ij} \tilde{\pi}_{ij}}{\omega} - \frac{3\omega \pi^2}{8} - \omega^{1/3} \tilde{R} + 2\omega \Lambda \]

By taking a square root, we can solve \( H_0 \sim 0 \) to obtain \( \pi_\omega \sim \pi_{GR} \).

\[ \pi_{GR} \equiv -\sqrt{\frac{8}{3}} \sqrt{\frac{\tilde{\pi}^{ij} \tilde{\pi}_{ij}}{\omega^2} - \frac{\tilde{R}}{\omega^{2/3}}} + 2\Lambda \]
Phase Space Reduction

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\[ \pi_{\text{GR}} \equiv -\sqrt{\frac{8}{3}} \sqrt{\frac{\tilde{\pi}^{ij} \tilde{\pi}_{ij}}{\omega^2} - \frac{\tilde{R}}{\omega^{2/3}}} + 2\Lambda \]

Expanding

\[ (h_{ij}, \pi^{ij}) \quad (\mathcal{H}_0, \chi) \rightarrow (\omega, \pi_\omega), (\tilde{h}_{ij}, \tilde{\pi}^{ij}) \]
Spatially Covariant GR

\[ S = \int dt \, d^3x \left( \tilde{h}_{ij} \tilde{\pi}^{ij} + \dot{\omega} \pi_\omega - N^i \tilde{\mathcal{H}}_i \right) \]

\[ \pi_\omega = \pi_{\text{GR}} \quad \tilde{\mathcal{H}}_i = -2 \tilde{h}_{ij} \tilde{\nabla}^k \tilde{\pi}^{jk} - \omega \tilde{\nabla}^i \pi_\omega \]

\( N^i \)’s are Lagrange multipliers enforcing \( \tilde{\mathcal{H}}_i \approx 0 \)

\[ \dot{A} = \frac{\partial A}{\partial t} + \{ A, H \} \quad H = \int d^3x \left( -\omega \pi_\omega + N^i \tilde{\mathcal{H}}_i \right) \]

Physical Hamiltonian Density

\[ \{ A, B \} \equiv \int d^3z \left( \frac{\delta A}{\delta \tilde{h}_{mn}(z)} \frac{\delta B}{\delta \tilde{\pi}^{mn}(z)} - \frac{\delta A}{\delta \tilde{\pi}^{mn}(z)} \frac{\delta B}{\delta \tilde{h}_{mn}(z)} \right) \]
Degrees of Freedom

Phase Space: \( (\tilde{h}_{ij}, \tilde{\pi}^{ij}) \)

First Class Constraints: \( \tilde{\mathcal{H}}_i \) \[ \{ \tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_j(y) \} \sim 0 \]

Arbitrary Functions: \( N^i \)

\[ 5 \tilde{h}_{ij}'s + 5 \tilde{\pi}^{ij}'s - 3 \tilde{\mathcal{H}}_i's - 3 N^i's \]

\[ = 4 \text{ canonical degrees of freedom} \]

Counting depends solely on

- Size of phase space
- First class constraint structure
Modified Gravity

Modify the physical Hamiltonian density on the reduced phase space \( \tilde{h}_{ij}, \tilde{\pi}^{ij} \)

Focus on \( \tilde{\pi}_\omega \), the scalar part of the physical Hamiltonian density

To represent spatial covariance and preserve counting of degrees of freedom, we demand two conditions:

First Class Algebra

\[ \{ \tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_j(y) \} \sim 0 \]

Consistency

\[ \dot{\tilde{\mathcal{H}}}_i \sim 0 \]

(Spatially covariant GR satisfies both)
Modified Gravity

Modify the physical Hamiltonian density on the reduced phase space $\tilde{h}_{ij}$, $\tilde{\pi}^{ij}$

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To represent spatial covariance and preserve counting of degrees of freedom, we demand two conditions:

**First Class Algebra**

$$\{\tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_j(y)\} \sim 0$$

**Consistency**

$$\dot{\tilde{\mathcal{H}}}_i \sim 0$$

(Spatially covariant GR satisfies both)

What freedom is there to modify $\pi_\omega$?
Gradient Expansion

Expand in powers of spatial derivatives $\tilde{\nabla}_i$

$$\pi_\omega = \pi_0 + \pi_1 (\tilde{\nabla}_i) + \pi_2 (\tilde{\nabla}_i \tilde{\nabla}_j) + \ldots$$

Aside from the momentum constraints $\tilde{\mathcal{H}}_i$, there are no vector quantities, so $\pi_1$ vanishes.

Taking only the lowest order term yields an ultralocal theory of gravity.
Assume $\pi_\omega$ contains no spatial derivatives; this is a long-wavelength, deep \textbf{infrared} limit in which gravitons have only \textbf{kinetic} energy.

Preserve form of momentum constraints

$$\tilde{\mathcal{H}}_i = -2\tilde{h}_{ij} \tilde{\nabla}_k \tilde{\pi}^{jk} - \omega \tilde{\nabla}_i \pi_\omega$$

Most general $\pi_\omega$ is an arbitrary function of time $t$ and the scalars $\phi(n)$

$$\phi(n) \equiv \tilde{\pi}^{i_n}_{i_1} \tilde{\pi}^{i_2}_{i_1} \ldots \tilde{\pi}^{i_{n-1}}_{i_n}$$
Computing the Algebra

Tensor part
\[ \mathcal{J}_i \equiv -2\tilde{h}_{ij} \tilde{\nabla}_k \tilde{\pi}^{jk} \]

Scalar Part
\[ \tilde{\mathcal{H}}_i = \mathcal{J}_i + \mathcal{K}_i \]
\[ \mathcal{K}_i \equiv -\omega \tilde{\nabla}_i \pi \omega \]

Variations of \( \mathcal{J}_i, \mathcal{K}_i \) involve spatial derivatives acting on the field variations, complicating the Poisson brackets.

To compute Poisson brackets, introduce the smoothing functions \( f^i(x), g^a(y) \), and compute Poisson brackets of the smoothing functionals

\[ F_J \equiv \int d^3x f^i \mathcal{J}_i \quad F_K \equiv \int d^3x f^i \mathcal{K}_i \]
\[ G_J \equiv \int d^3y g^a \mathcal{J}_a \quad G_K \equiv \int d^3y g^a \mathcal{K}_a \]

Derive distributional identities, i.e., identities which hold “for all \( f^i(x), g^a(y) \)”
Sample Bracket

\[ F_J \equiv \int d^3 x f^i J_i \quad G_J \equiv \int d^3 y g^a J_a \]

\[ \{ F_J, G_J \} = \int d^3 x d^3 y f^i(x) g^a(y) [ J_i(x), J_a(y) ] \]

\[ \frac{\delta F_J}{\delta h_{mn}} = 2 \tilde{\delta}_{ij}^{mn} \tilde{\pi}^{jk} \tilde{\nabla}_k f^i - \tilde{\nabla}_i \left( f^i \tilde{\pi}^{mn} \right) - \frac{2}{3} \tilde{\pi}^{mn} \tilde{\nabla}_i f^i \quad \frac{\delta F_J}{\delta \tilde{\pi}^{mn}} = 2 \delta_{jk}^{mn} \tilde{\pi}^{ij} \tilde{\nabla}_k f^i \]

\[ \{ F_J, G_J \} = 2 \int d^3 z \left\{ \left( \tilde{\nabla}_c f^i \right) \left( \tilde{\nabla}_i g^a \right) \tilde{h}_{ab} \tilde{\pi}^{bc} - \left( \tilde{\nabla}_k g^a \right) \left( \tilde{\nabla}_a f^i \right) \tilde{h}_{ij} \tilde{\pi}^{jk} + \left( \tilde{\nabla}_k f^i \right) \tilde{\nabla}_a \left( g^a \tilde{h}_{ij} \tilde{\pi}^{jk} \right) - \left( \tilde{\nabla}_c g^a \right) \tilde{\nabla}_i \left( f^i \tilde{h}_{ab} \tilde{\pi}^{bc} \right) \right\} \]

\[ \{ F_J, G_J \} = \int d^3 x d^3 y f^i(x) g^a(y) \left( J_a(x) \partial_{x^i} \delta^3(x - y) - J_i(y) \partial_{y^a} \delta^3(x - y) \right) \]

\[ \{ J_i(x), J_a(y) \} = J_a(x) \partial_{x^i} \delta^3(x - y) - J_i(y) \partial_{y^a} \delta^3(x - y) \]
The Ultralocal Algebra

\[ \tilde{H}_i = J_i + K_i \]

Tensor part

\[ J_i \equiv -2\tilde{h}_{ij} \tilde{\nabla}_k \tilde{\pi}^{jk} \]

Scalar Part

\[ K_i \equiv -\omega \tilde{\nabla}_i \pi_\omega \]

\[
\begin{align*}
\{J_i(x), J_a(y)\} &= J_a(x)\partial_{x^i} \delta^3(x - y) - J_i(y)\partial_{y^a} \delta^3(x - y) \\
\{J_i(x), K_a(y)\} + \{K_i(x), J_a(y)\} &= K_a(x)\partial_{x^i} \delta^3(x - y) - K_i(y)\partial_{y^a} \delta^3(x - y) \\
\{K_i(x), K_a(y)\} &= 0 \\
\{\tilde{H}_i(x), \tilde{H}_j(y)\} &= \tilde{H}_j(x)\partial_{x^i} \delta^3(x - y) - \tilde{H}_i(y)\partial_{y^j} \delta^3(x - y) \\
\{\tilde{H}_i(x), \tilde{H}_j(y)\} &\sim 0
\end{align*}
\]
By assumption, \( \omega(t) \) is invertible, so we can take \( \pi_\omega \) to depend on \( \omega \) and 
\[
\phi(n) \equiv \tilde{\pi}^{i_1} \tilde{\pi}^{i_2} \ldots \tilde{\pi}^{i_{n-1}}
\]

The constraints are first class, so

\[
\dot{\mathcal{H}}_i(x) = -\omega \partial_{x^i} \Delta \pi_\omega(x) + \int d^3 y \; N^j(y) \{ \tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_j(y) \}
\]


\[
\Delta \equiv \omega \frac{\partial}{\partial \omega} + \sum_{m=2}^{\infty} m \phi(m) \frac{\partial}{\partial \phi(m)}
\]

The constraints are first class, so

\[
\dot{\mathcal{H}}_i \sim 0 \quad \rightarrow \quad \Delta \pi_\omega = 0
\]

This is a conformal RG equation!
Conformal Symmetry

- Spatial Conformal Scaling
  \[ x^i \rightarrow x^i / \lambda \]
  \[ \omega \rightarrow \lambda^3 \cdot \omega \quad \tilde{\pi}^i_j \rightarrow \lambda^3 \cdot \tilde{\pi}^i_j \]

- The invariant scalars are
  \[ \bar{\phi}(n) \equiv \frac{\phi(n)}{\omega^n} \quad \phi(n) \equiv \tilde{\pi}^{i_{n-1}}_{i_n} \tilde{\pi}^{i_{n-2}}_{i_{n-1}} \cdots \tilde{\pi}^{i_1}_{i_2} \]

- The most general solution to \( \Delta \pi_\omega = 0 \) is an arbitrary function of the \( \bar{\phi}(n) \)
  \[ \Delta \pi_\omega = 0 \rightarrow \pi_\omega (\bar{\phi}(2), \bar{\phi}(3), \ldots) \]
Ultralocal Summary

- The ultralocal Hamiltonian must obey a conformal RG equation:

\[ 0 = \left( \omega \frac{\partial}{\partial \omega} + \sum_{m=2}^{\infty} m \phi(m) \frac{\partial}{\partial \phi(m)} \right) \pi \omega \]

- This condition encodes invariance under flow through the space of conformally equivalent spatial metrics.

- That’s it! There are no other restrictions.
Preliminary Conclusions

Can we modify the behavior of the graviton?
Can we modify GR without new DOF?
Preliminary Conclusions

Can we modify the behavior of the graviton? Can we modify GR \textit{without} new DOF?

YES!
Preliminary Conclusions

Can we modify the behavior of the graviton?
Can we modify GR without new DOF?

YES!

Yes, provided the theory is invariant under conformal RG flow.

General relativity is not the unique theory of a transverse, traceless graviton.
Local Case

Allow $\pi_\omega$ to depend on spatial derivatives through the Ricci scalar $\tilde{R}$ of $\tilde{h}_{ij}$.

Ricci scalar dependence is the **leading local correction** to infrared dynamics.

Most general $\pi_\omega$ is an arbitrary function of time $t$, $\tilde{R}$, and the scalars $\phi(n) = \tilde{\pi}^{i_1}_{i_1} \tilde{\pi}^{i_2}_{i_2} \ldots \tilde{\pi}^{i_{n-1}}_{i_{n}}$.

This class of theories **includes** GR.

$$\pi_{GR} = -\sqrt{\frac{8}{3}} \sqrt{\frac{\phi(2)}{\omega^2}} - \frac{\tilde{R}}{\omega^{2/3}} + 2\Lambda$$
Local Case

- Allow $\pi_\omega$ to depend on spatial derivatives through the Ricci scalar $\tilde{R}$ of $\tilde{h}_{ij}$.

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- This class of theories includes GR:

\[
\pi_{\text{GR}} = -\sqrt{\frac{8}{3}} \sqrt{\phi(2) \frac{\omega^2}{\omega^{2/3}}} - \frac{\tilde{R}}{\omega^{2/3}} + 2\Lambda
\]

How does locality constrain the physical Hamiltonian?
Local Consistency

Assuming the constraints $\mathcal{H}_i$ are first class,

$$\dot{\mathcal{H}}_i \sim 0 \quad \Rightarrow \quad \Delta \pi_\omega = 0$$

$$\Delta \equiv \omega \frac{\partial}{\partial \omega} + \frac{2}{3} \tilde{R} \frac{\partial}{\partial \tilde{R}} + \sum_{m=2}^{\infty} m \phi(m) \frac{\partial}{\partial \phi(m)}$$

Conformal Scaling

$$\omega \rightarrow \mu \cdot \omega \quad \tilde{R} \rightarrow \mu^{2/3} \cdot \tilde{R}$$

Invariant scalars are the $\phi(n)$ and

$$\bar{R} \equiv \frac{\tilde{R}}{\omega^{2/3}}$$

Most general solution is a function of $R$ and the $\phi(n)$
Local Algebra

Tensor part
\[ J_i \equiv -2\tilde{\hbar}_{ij} \tilde{\nabla}_k \tilde{\pi}^{jk} \]

Scalar Part
\[ \tilde{\mathcal{H}}_i = J_i + \mathcal{K}_i \]
\[ \mathcal{K}_i \equiv -\omega \tilde{\nabla}_i \pi_\omega \]

As before,
\[ \{ J_i(x), J_a(y) \} = J_a(x) \partial_x^i \delta^3(x - y) - J_i(y) \partial_y^a \delta^3(x - y) \]

Poisson brackets involving \( \mathcal{K}_i \) are **MUCH more complicated in the realistic case**
\[ \delta \tilde{\mathcal{R}} = -\tilde{\mathcal{R}}^{jk} \delta \tilde{\hbar}_{jk} + \tilde{\nabla}^k \tilde{\nabla}^j \delta \tilde{\hbar}_{jk} \]

To compute these brackets, we resort once again to smoothing functionals
Sample Bracket

\[ F_K \equiv \int d^3 x \, f^i K_i \]
\[ G_J \equiv \int d^3 y \, g^a J_i \]

\[ \Pi^{ij}(n + 1) \equiv \tilde{\Pi}^i_k \Pi^{kj}(n) \]
\[ \Pi^{ij}(0) \equiv \tilde{h}^{ij} \]

\[ \frac{\delta F_K}{\delta \tilde{h}_{mn}} = \omega \left( \partial_i f^i \right) \sum_{n=2}^\infty n \frac{\partial \pi_\omega}{\partial \phi(n)} \left( \tilde{\pi}^{mn}_{jk} \Pi(n)_{jk} - \frac{1}{3} \tilde{\pi}^{mn}_\phi(n - 1) \right) \]

\[ - \omega \left( \partial_i f^i \right) \frac{\partial \pi_\omega}{\partial \tilde{R}} \tilde{\delta}^{mn}_{jk} \tilde{R}^k_{jk} + \omega \tilde{\delta}^{mn}_{jk} \tilde{\nabla}_j \tilde{\nabla}^k \left( \partial_i f^i \right) \frac{\partial \pi_\omega}{\partial \tilde{R}} \]

\[ \frac{\delta F_K}{\delta \tilde{\pi}^{mn}} = \omega \left( \partial_i f^i \right) \sum_{n=2}^\infty n \frac{\partial \pi_\omega}{\partial \phi(n)} \tilde{\delta}^{jk}_{mn} \Pi(n - 1)_{jk} \]

\[ \{ F_J, G_K \} = -\omega \int d^3 z \, f^i \left( \partial_a g^a \right) \sum_{m=2}^\infty \frac{\partial \pi_\omega}{\partial \phi(m)} \tilde{\nabla}_i \phi(m) + 2\omega \int d^3 z \, \left( \partial_a g^a \right) \frac{\partial \pi_\omega}{\partial \tilde{R}} \tilde{R}^k_{ij} \tilde{\nabla}_k f^i \]

\[ -2\omega \int d^3 z \, \left( \tilde{\nabla}_k f^i \right) \tilde{\nabla}_i \tilde{\nabla}^k \left( \left( \partial_a g^a \right) \frac{\partial \pi_\omega}{\partial \tilde{R}} \right) + \frac{2}{3} \omega \int d^3 z \, \left( \partial_i f^i \right) \tilde{\nabla}_c \tilde{\nabla}^c \left( \left( \partial_a g^a \right) \frac{\partial \pi_\omega}{\partial \tilde{R}} \right) \]

\[ -\omega \int d^3 z \, \left( \partial_i f^i \right) \left( \partial_a g^a \right) \left( \frac{2}{3} \tilde{R} \frac{\partial \pi_\omega}{\partial \tilde{R}} + \sum_{m=2}^\infty m \phi(m) \frac{\partial \pi_\omega}{\partial \phi(m)} \right) \]
\[ \{F_J, G_K\} = \int d^3z f^i \mathcal{K}_i \partial_a g^a + \frac{4}{3} \omega \int d^3z \tilde{\nabla}_k (\partial_if^i) \tilde{\nabla}^k \left( (\partial_ag^a) \frac{\partial \pi_\omega}{\partial \tilde{R}} \right) \]
\[ -\omega \int d^3z (\partial_if^i) (\partial_ag^a) \left( \frac{2}{3} \tilde{R} \frac{\partial \pi_\omega}{\partial \tilde{R}} + \sum_{m=2}^{\infty} m\phi(m) \frac{\partial \pi_\omega}{\partial \phi(m)} \right) \]
\[ \{F_K, G_J\} = -\int d^3z g^a \mathcal{K}_a \partial_i f^i - \frac{4}{3} \omega \int d^3z \tilde{\nabla}_k (\partial_ag^a) \tilde{\nabla}^k \left( (\partial_if^i) \frac{\partial \pi_\omega}{\partial \tilde{R}} \right) \]
\[ +\omega \int d^3z (\partial_if^i) (\partial_ag^a) \left( \frac{2}{3} \tilde{R} \frac{\partial \pi_\omega}{\partial \tilde{R}} + \sum_{m=2}^{\infty} m\phi(m) \frac{\partial \pi_\omega}{\partial \phi(m)} \right) \]
\[ \{F_J, G_K\} + \{F_K, G_J\} = \int d^3z f^i \mathcal{K}_i \partial_a g^a - \int d^3z g^a \mathcal{K}_a \partial_i f^i \]
\[ + \frac{4}{3} \omega \int d^3z \tilde{\nabla}_k (\partial_if^i) \tilde{\nabla}^k \left( (\partial_ag^a) \frac{\partial \pi_\omega}{\partial \tilde{R}} \right) \]
\[ - \frac{4}{3} \omega \int d^3z \tilde{\nabla}_k (\partial_ag^a) \tilde{\nabla}^k \left( (\partial_if^i) \frac{\partial \pi_\omega}{\partial \tilde{R}} \right) \]
\[
\{ \tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_j(y) \} = \tilde{\mathcal{H}}_j(x) \partial_{x^i} \delta^3(x - y) - \tilde{\mathcal{H}}_i(y) \partial_{y^j} \delta^3(x - y) \\
+ \partial_{x^i} \left( -I^k(x) \partial_{x^k} \partial_{x^j} \delta^3(x - y) \right) - \partial_{y^j} \left( -I^k(y) \partial_{y^k} \partial_{y^i} \delta^3(x - y) \right)
\]

\[
I_k \equiv \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{R}} \tilde{\nabla}^j \frac{\partial \pi_\omega}{\partial \tilde{\pi}^{jk}} - \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{\pi}^{jk}} \tilde{\nabla}^j \frac{\partial \pi_\omega}{\partial \tilde{R}} - \frac{4}{3} \omega \tilde{\nabla}^k \frac{\partial \pi_\omega}{\partial \tilde{R}}
\]

\[
\{ \tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_j(y) \} \sim 0 \quad \Rightarrow \quad I_k \sim 0
\]

- Ultralocal Case \( I_k = 0 \)
- Spatially Covariant GR \( I_k(\pi_{GR}) = \frac{16}{9\omega^{2/3} \pi_{GR}^2} \mathcal{H}_k \)
\[
\{ \tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_j(y) \} = \tilde{\mathcal{H}}_j(x) \partial_{x^i} \delta^3(x - y) - \tilde{\mathcal{H}}_i(y) \partial_{y^j} \delta^3(x - y) \\
+ \partial_{x^i} \left( -I^k(x) \partial_{x^k} \partial_{x^j} \delta^3(x - y) \right) - \partial_{y^j} \left( -I^k(y) \partial_{y^k} \partial_{y^i} \delta^3(x - y) \right)
\]

\[
I_k \equiv \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{R}} \nabla^j \frac{\partial \pi_\omega}{\partial \tilde{\pi}^{jk}} - \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{\pi}^{jk}} \nabla^j \frac{\partial \pi_\omega}{\partial \tilde{R}} - \frac{4}{3} \omega \tilde{\nabla}^k \frac{\partial \pi_\omega}{\partial \tilde{R}}
\]

\[
\{ \tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_j(y) \} \sim 0 \quad \Rightarrow \quad I_k \sim 0
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- Ultralocal Case \( I_k = 0 \)
- Spatially Covariant GR \( I_k(\pi_{GR}) = \frac{16}{9 \omega^{2/3} \pi_{GR}^2} \mathcal{H}_k \)

This is highly non-trivial!
\[ I_k \equiv \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{R}} \tilde{\nabla}^j \frac{\partial \pi_\omega}{\partial \tilde{\pi}^{jk}} - \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{\pi}^{jk}} \tilde{\nabla}^j \frac{\partial \pi_\omega}{\partial \tilde{R}} - \frac{4}{3} \omega \tilde{\nabla}^k \frac{\partial \pi_\omega}{\partial \tilde{R}} \]
\[ \mathcal{I}_k \equiv \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{\nabla}_R} \tilde{\nabla}_j \frac{\partial \pi_\omega}{\partial \tilde{\pi}^{jk}} - \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{\pi}^{jk}} \tilde{\nabla}_j \frac{\partial \pi_\omega}{\partial \tilde{\nabla}_R} - \frac{4}{3} \omega \tilde{\nabla}_k \frac{\partial \pi_\omega}{\partial \tilde{\nabla}_R} \]

\[
\pi_{GR} = -\sqrt{\frac{8}{3}} \sqrt{\frac{\phi(2)}{\omega^2}} - \frac{\tilde{R}}{\omega^{2/3}} + 2\Lambda
\]
\[ I_k \equiv \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{R}} \tilde{\nabla}^j \frac{\partial \pi_\omega}{\partial \tilde{\pi}^j k} - \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{\pi}^j k} \tilde{\nabla}^j \frac{\partial \pi_\omega}{\partial \tilde{R}} - \frac{4}{3} \omega \tilde{\nabla}^k \frac{\partial \pi_\omega}{\partial \tilde{R}} \]

\[ \pi_{GR} = -\sqrt{\frac{8}{3}} \sqrt{\frac{\phi(2)}{\omega^2}} - \frac{\tilde{R}}{\omega^{2/3}} + 2\Lambda \]

\[ I_k(\pi_{GR}) = -\frac{4}{3} \omega \tilde{\nabla}^k \frac{\partial \pi_{GR}}{\partial \tilde{R}} - \omega^2 \frac{\partial \pi_{GR}}{\partial \tilde{R}} \frac{\partial \pi_{GR}}{\partial \phi(2)} J_k + 2\omega^2 \tilde{\pi}^j k \left( \frac{\partial \pi_{GR}}{\partial \tilde{R}} \tilde{\nabla}^j \frac{\partial \pi_{GR}}{\partial \phi(2)} - \frac{\partial \pi_{GR}}{\partial \phi(2)} \tilde{\nabla}^j \frac{\partial \pi_{GR}}{\partial \tilde{R}} \right) \]
\[ \mathcal{I}_k \equiv \omega^2 \frac{\partial \pi}{\partial \tilde{R}} \tilde{\nabla}^j \frac{\partial \pi}{\partial \tilde{\pi}^{jk}} - \omega^2 \frac{\partial \pi}{\partial \tilde{\pi}^{jk}} \tilde{\nabla}^j \frac{\partial \pi}{\partial \tilde{R}} - \frac{4}{3} \omega \tilde{\nabla}_k \frac{\partial \pi}{\partial \tilde{R}} \]

\[
\pi_{GR} = -\sqrt{\frac{8}{3}} \sqrt{\frac{\phi(2)}{\omega^2}} - \frac{\tilde{R}}{\omega^{2/3}} + 2\Lambda
\]

\[
\mathcal{I}_k(\pi_{GR}) = -\frac{4}{3} \omega \tilde{\nabla}_k \frac{\partial \pi_{GR}}{\partial \tilde{R}} - \omega^2 \frac{\partial \pi_{GR}}{\partial \tilde{R}} \frac{\partial \pi_{GR}}{\partial \phi(2)} \mathcal{J}_k + 2\omega^2 \tilde{\pi}^{jk} \left( \frac{\partial \pi_{GR}}{\partial \tilde{R}} \tilde{\nabla}^j \frac{\partial \pi_{GR}}{\partial \phi(2)} - \frac{\partial \pi_{GR}}{\partial \phi(2)} \tilde{\nabla}^j \frac{\partial \pi_{GR}}{\partial \tilde{R}} \right)
\]

\[
\mathcal{I}_k(\pi_{GR}) = -\frac{4}{3} \omega \tilde{\nabla}_k \frac{\partial \pi_{GR}}{\partial \tilde{R}} - \omega^2 \frac{\partial \pi_{GR}}{\partial \tilde{R}} \frac{\partial \pi_{GR}}{\partial \phi(2)} \mathcal{J}_k
\]
\[ I_k \equiv \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{R}} \tilde{\nabla}^j j \frac{\partial \pi_\omega}{\partial \tilde{\pi}^j k} - \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{\pi}^j k} \tilde{\nabla}^j j \frac{\partial \pi_\omega}{\partial \tilde{R}} - \frac{4}{3} \omega \tilde{\nabla}_k \frac{\partial \pi_\omega}{\partial \tilde{R}} \]

\[ \pi_{GR} = -\sqrt{\frac{8}{3}} \sqrt{\frac{\phi(2)}{\omega^2}} - \frac{\tilde{R}}{\omega^{2/3}} + 2\Lambda \]

\[ I_k(\pi_{GR}) = -\frac{4}{3} \omega \tilde{\nabla}_k \frac{\partial \pi_{GR}}{\partial \tilde{R}} - \omega^2 \frac{\partial \pi_{GR}}{\partial \tilde{R}} \frac{\partial \pi_{GR}}{\partial \phi(2)} J_k + 2\omega^2 \tilde{\pi}^j_\cdot j \left( \frac{\partial \pi_{GR}}{\partial \tilde{R}} \tilde{\nabla}^j j \frac{\partial \pi_{GR}}{\partial \phi(2)} - \frac{\partial \pi_{GR}}{\partial \phi(2)} \tilde{\nabla}^j j \frac{\partial \pi_{GR}}{\partial \tilde{R}} \right) \]

\[ I_k(\pi_{GR}) = -\frac{4}{3} \omega \tilde{\nabla}_k \frac{\partial \pi_{GR}}{\partial \tilde{R}} - \omega^2 \frac{\partial \pi_{GR}}{\partial \tilde{R}} \frac{\partial \pi_{GR}}{\partial \phi(2)} J_k \]

\[ I_k(\pi_{GR}) = \frac{16}{9\omega^{2/3} \pi_{GR}^2} K_k + \frac{16}{9\omega^{2/3} \pi_{GR}^2} J_k \]
\[ \mathcal{I}_k \equiv \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{R}} \tilde{\nabla}_j \frac{\partial \pi_\omega}{\partial \tilde{\pi}^{jk}} - \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{\pi}^{jk}} \tilde{\nabla}_j \frac{\partial \pi_\omega}{\partial \tilde{R}} - \frac{4}{3} \omega \tilde{\nabla}_k \frac{\partial \pi_\omega}{\partial \tilde{R}} \]

\[ \pi_{\text{GR}} = -\sqrt{\frac{8}{3}} \sqrt{\frac{\phi(2)}{\omega^2}} - \frac{\tilde{R}}{\omega^{2/3}} + 2\Lambda \]

\[ \mathcal{I}_k(\pi_{\text{GR}}) = -\frac{4}{3} \omega \tilde{\nabla}_k \frac{\partial \pi_{\text{GR}}}{\partial \tilde{R}} - \omega^2 \frac{\partial \pi_{\text{GR}}}{\partial \tilde{R}} \frac{\partial \pi_{\text{GR}}}{\partial \phi(2)} \mathcal{J}_k + 2\omega^2 \tilde{\pi}^{jk} \left( \frac{\partial \pi_{\text{GR}}}{\partial \tilde{R}} \tilde{\nabla}_j \frac{\partial \pi_{\text{GR}}}{\partial \phi(2)} - \frac{\partial \pi_{\text{GR}}}{\partial \phi(2)} \tilde{\nabla}_j \frac{\partial \pi_{\text{GR}}}{\partial \tilde{R}} \right) \]

\[ \mathcal{I}_k(\pi_{\text{GR}}) = -\frac{4}{3} \omega \tilde{\nabla}_k \frac{\partial \pi_{\text{GR}}}{\partial \tilde{R}} - \omega^2 \frac{\partial \pi_{\text{GR}}}{\partial \tilde{R}} \frac{\partial \pi_{\text{GR}}}{\partial \phi(2)} \mathcal{J}_k \]

\[ \mathcal{I}_k(\pi_{\text{GR}}) = \frac{16}{9\omega^{2/3} \pi^2_{\text{GR}}} \mathcal{K}_k + \frac{16}{9\omega^{2/3} \pi^2_{\text{GR}}} \mathcal{J}_k \]

\[ \mathcal{I}_k(\pi_{\text{GR}}) = \frac{16}{9\omega^{2/3} \pi^2_{\text{GR}}} \mathcal{H}_k \]
Local theories obey a generalized version of the RG equation \( \Delta \pi_\omega = 0 \), which encodes invariance under flow through the space of conformally equivalent spatial metrics.

Local theories must also obey the differential condition \( I_k \sim 0 \), which is satisfied non-trivially by GR.

These two conditions are necessary and sufficient for local \( \pi_\omega \) to yield a consistent theory of the graviton.
Summary of Results

We can modify GR without new DOF, provided the theory is invariant under conformal scaling of the spatial metric.

In the ultralocal limit, $\pi_\omega$ can have arbitrary dependence on the invariant kinetic scalars $\bar{\phi}(n)$

$$\bar{\phi}(n) \equiv \frac{\phi(n)}{\omega^n} \quad \phi(n) \equiv \tilde{\pi}^{i_n}_{i_1} \tilde{\pi}^{i_1}_{i_2} \ldots \tilde{\pi}^{i_{n-1}}_{i_n}$$

At leading order, locality constrains $\pi_\omega$ through the consistency condition $I_k \sim 0$

$$I_k \equiv \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{\nabla}^j} \frac{\partial \pi_\omega}{\partial \tilde{\pi}^{j_k}} - \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{\pi}^{j_k}} \frac{\partial \pi_\omega}{\partial \tilde{\nabla}^j} - \frac{4}{3} \omega \tilde{\nabla}^k \frac{\partial \pi_\omega}{\partial \tilde{\nabla}^k}$$
Study solutions to $I_k \sim \Omega$, focusing on solutions perturbed around $\pi_\omega = \pi_{\text{GR}}$.

Is it possible to modify general relativity parametrically in the infrared?

Generalize results to include a possible dependence of $\pi_\omega$ on more general derivative quantities, e.g.,

$$\tilde{R}_{ij} \tilde{R}^{ij}, \tilde{R}_{ij} \tilde{\pi}^{ij}, \tilde{\nabla}_k \tilde{\nabla}_k \tilde{\pi}^{ij}, \ldots$$

How does locality constrain the infrared behavior of the graviton?
Win-Win Situation

Is GR the unique low-energy local, realistic theory of the graviton degrees of freedom?

If so, Lorentz invariance in the gravitational sector arises as an accidental symmetry.

If not, infrared modifications of the graviton could shed light on dark energy.

Either way, the results will be interesting!
Win-Win Situation

- Is GR the unique low-energy local, realistic theory of the graviton degrees of freedom?
- If so, Lorentz invariance in the gravitational sector arises as an accidental symmetry.
- If not, infrared modifications of the graviton could shed light on dark energy.
- Either way, the results will be interesting!

Thank you