

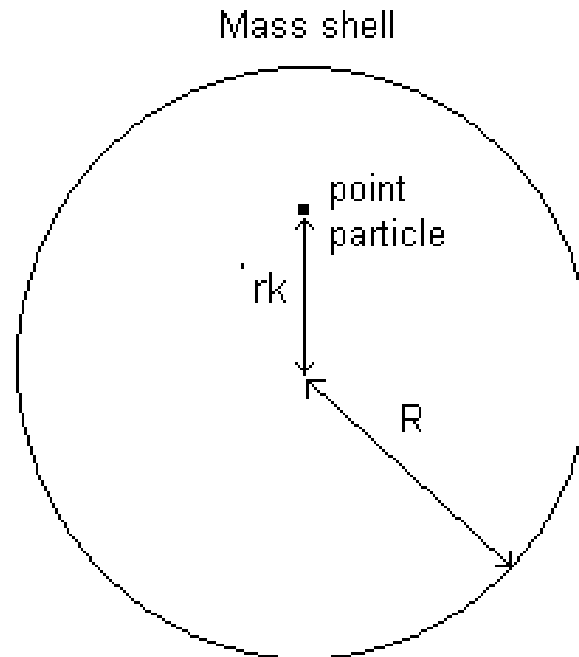
Does Birkhoff's law hold in  
MOND?

# Birkhoff's Theorem

- Any spherically symmetric solution of the Einstein field equations in vacuum must be stationary and asymptotically flat.
- An important corollary is that the metric inside a spherical shell is the Minkowski metric.
- The Newtonian counterpart of the theory is that the gravitational field inside a spherical cavity vanishes.
- Birkhoff's theorem reduces the computational labor when the mass distribution is spherically symmetric.
- Does the theorem hold approximately when the mass distribution is approximately spherically symmetric?

- The spherical symmetry is perturbed by the finite mass test particle

$M_s$ - mass of the shell  
 $M_k$ - mass of the test particle



# Newtonian gravity

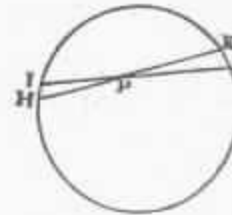
- In Newtonian gravity, the theorem holds true even when the test particle is massive.

## SECTION XII

*Of the attractive forces of spherical bodies.*

### PROPOSITION LXX. THEOREM XXX.

*If to every point of a spherical surface there tend equal centripetal forces decreasing in the duplicate ratio of the distances from those points; I say, that a corpuscle placed within that superficies will not be attracted by those forces any way.*



Let HIKL, be that spherical superficies, and P a corpuscle placed within. Through P let there be drawn to this superficies to two lines HK, IL, intercepting very small arcs HI, KL; and because (by Cor. 3, Lem. VII) the triangles HPI, LPK are alike, those arcs will be proportional to the distances HP, LP; and any particles at HI and KL of the spherical superficies, terminated by right lines passing through P, will be in the duplicate ratio of those distances. Therefore the forces of these particles exerted upon the body P are equal between themselves. For the forces are as the particles directly, and the squares of the distances inversely. And these two ratios compose the ratio of equality. The attractions therefore, being made equally towards contrary parts, destroy each other. And by a like reasoning all the attractions through the whole spherical superficies are destroyed by contrary attractions. Therefore the body P will not be any way impelled by those attractions. Q.E.D.

# General Relativity

- Since the massive test particle distorts the space around it, a “spherical shell” ceases to be a spherical shell.
- The spherical shell is defined using Parametrized Post-Newtonian (PPN) coordinate. The center is defined to be a point which has the same distance to each point on the shell in PPN coordinate.
- Using EIH equation, in the limit  $M_s \gg M_k$  and  $r_k \ll R$ , the acceleration on the test particle is found to be

$$a_k = -g_{Ns} \frac{M_k}{M_s} \frac{R_{Schs}}{R} \frac{1}{2d} \left[ \frac{1}{1-d^2} + \frac{1}{2d} \ln \left( \frac{1-d}{1+d} \right) \right] \hat{z}$$

$$\approx -g_{Ns} \frac{M_k}{M_s} \frac{R_{Schs}}{R} \left( \frac{1}{3}d + \frac{2}{5}d^3 + \frac{3}{7}d^5 + \dots \right) \hat{z}$$

Defining  $d = \frac{r_k}{R}$ ,  $R_{Schs} = 2GM_s$ , and  $g_{Ns} = \frac{GM_s}{R^2}$

This is a small number.

# MOND

- MOND is an empirically-based modification of Newtonian gravity. It is proposed to explain galactic rotation curves.
- MOND deviate from Newtonian gravity when the field is weak compared to MOND scale  $a_0$  (typically occurs at large distance). The MOND does not follow inverse square law in this regime.

- MOND assumes the modified form of Poisson equation.

$$\nabla \cdot \left( \mu \left( \frac{|\nabla\Phi|}{a_0} \right) \nabla\Phi \right) = 4\pi G\rho$$

- The function  $\mu$  has the property such that

$$\mu \rightarrow 1 \quad \text{for} \quad \left| \frac{\nabla\Phi}{a_0} \right| \gg 1$$

and

$$\mu \rightarrow \left| \frac{\nabla\Phi}{a_0} \right| \quad \text{for} \quad \left| \frac{\nabla\Phi}{a_0} \right| \ll 1$$

- We assume the specific form of  $\mu$  function.

$$\mu(x) = \frac{x}{\sqrt{1+x^2}}$$

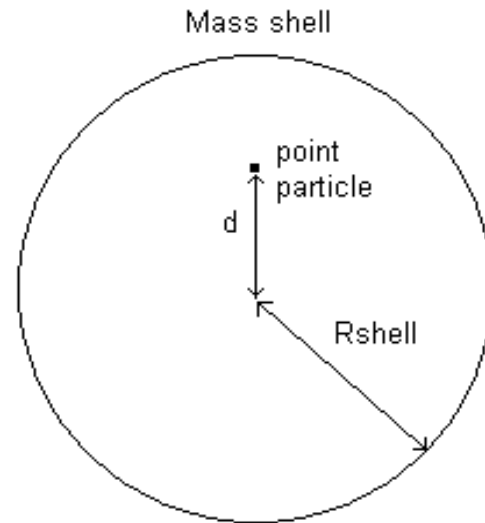
where

$$x = |\nabla\Phi / a_0|$$

$$a_0 = 1.2 \times 10^{-8} \text{ cm s}^{-2}$$



- System



- Parameters

Total mass =  $2.0 \times 10^{14}$  solar masses

Rshell = 2.0 Mpc

$$r_t = \sqrt{GM_{total} / a_0} = 0.48 \text{Mpc}$$

- In the limit  $M_p \gg M_{\text{shell}}$ , the modified Poisson equation can be solved perturbatively near the mass shell. Then, one can apply Newton's third law to find acceleration on the point particle. The result is (up to the first correction term from  $\rho_{\text{shell}}$ ),

$$\begin{aligned}
 \frac{a_{\text{particle}}}{a_0} &= \frac{M_{\text{shell}}}{M_{\text{particle}}} \cdot \frac{r_t}{R_{\text{shell}}} \cdot \frac{1}{2d} \cdot \left( 1 + \frac{1-d^2}{2d} \ln \left( \frac{1-d}{1+d} \right) \right) \\
 &\quad - \frac{1}{8} \cdot \frac{M_{\text{shell}}}{M_{\text{particle}}} \cdot \left( \frac{r_t}{R_{\text{shell}}} \right)^3 \cdot \frac{1}{d} \cdot \left( \frac{1}{1-d^2} + \frac{1}{2d} \ln \left( \frac{1-d}{1+d} \right) \right) \\
 &\quad - \frac{5}{24} \left( \frac{M_{\text{shell}}}{M_{\text{particle}}} \right)^2 \cdot \frac{r_t}{R_{\text{shell}}} \cdot d \quad \left( d = \frac{r_k}{R_{\text{shell}}} \right)
 \end{aligned}$$

- When  $M_p \gg M_{\text{shell}}$  is not true, the numerical simulation is used to find the acceleration on the point particle.
- We adopt a code developed by Milgrom (Milgrom 1985)
- A spherical lattice is used in the simulation, and the center of the lattice is placed at the position of the point particle.
- The code is tested in the regime  $M_p \gg M_{\text{shell}}$  where perturbative solution can be found, and the numerical values are compared to the analytical values.

## Little bit about the code

- We define the field  $U$  to be

$$\vec{U} \equiv \mu(|\nabla\phi|/a_0)\nabla\phi$$

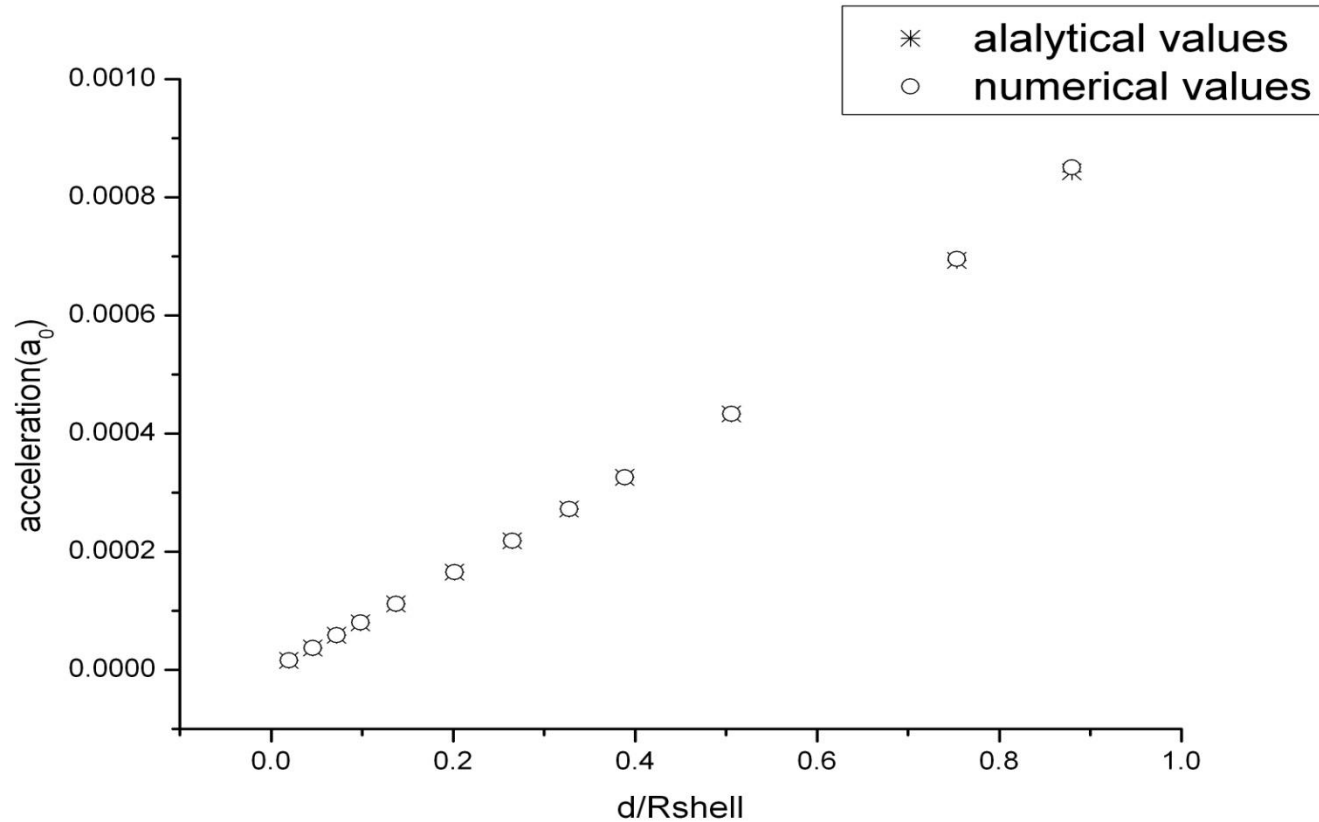
- For a monotonic  $\mu$  function, one can invert the equation

$$\nabla\phi = \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{a_0^2}{|U|^2}}} \vec{U}$$

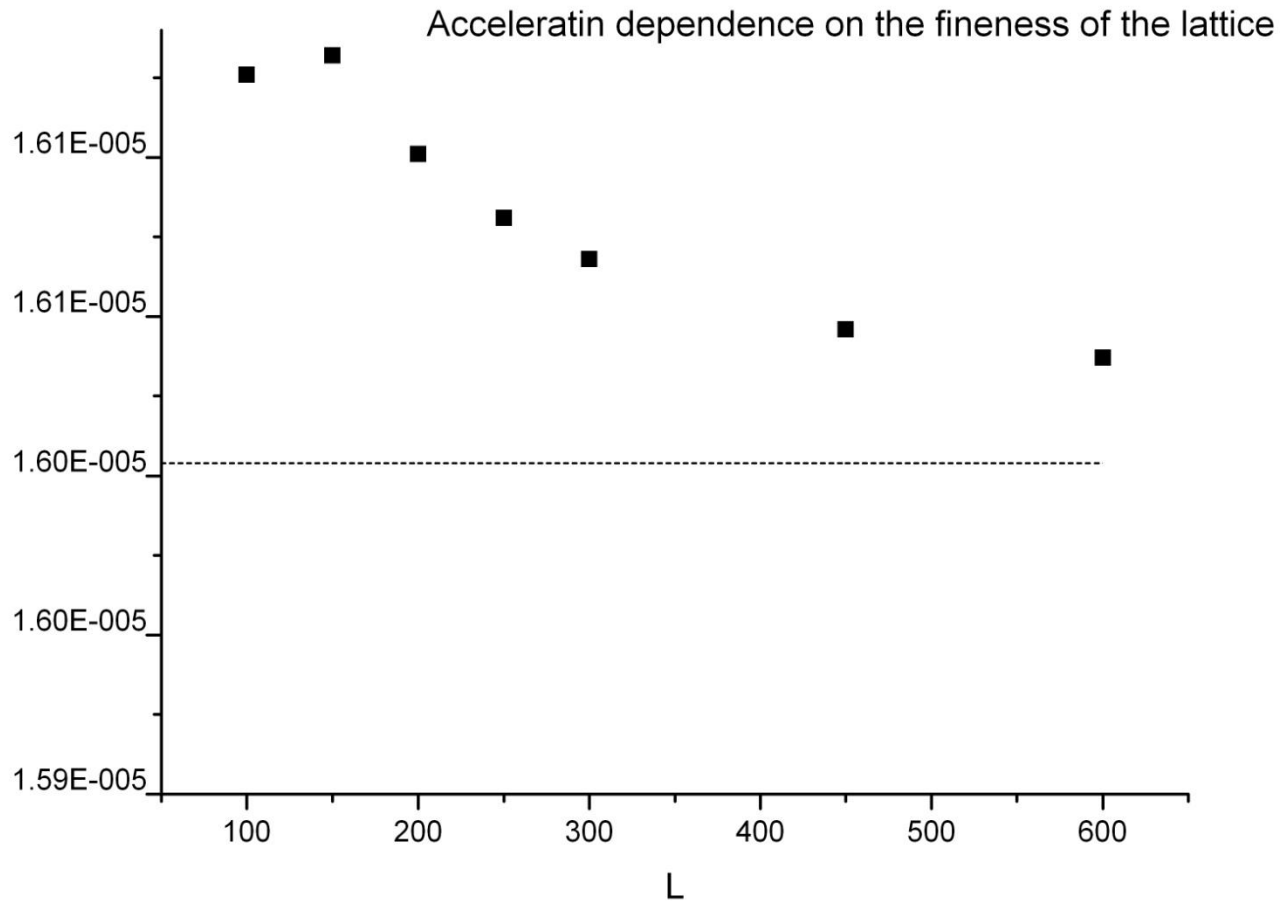
- The field  $U$  satisfies the set of differential equations

$$\vec{\nabla} \cdot \vec{U} = 4\pi G\rho \quad \text{and} \quad \nabla \times \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{a_0^2}{|U|^2}}} \vec{U} = 0$$

- The numerical calculation is carried out in terms of  $U$ , and the initial condition as well as the boundary condition is given by the solution of the first differential equation.



The difference is at most .9%, and larger discrepancy occur for larger k.  
(This is for  $M_s/M_k=0.01$ )



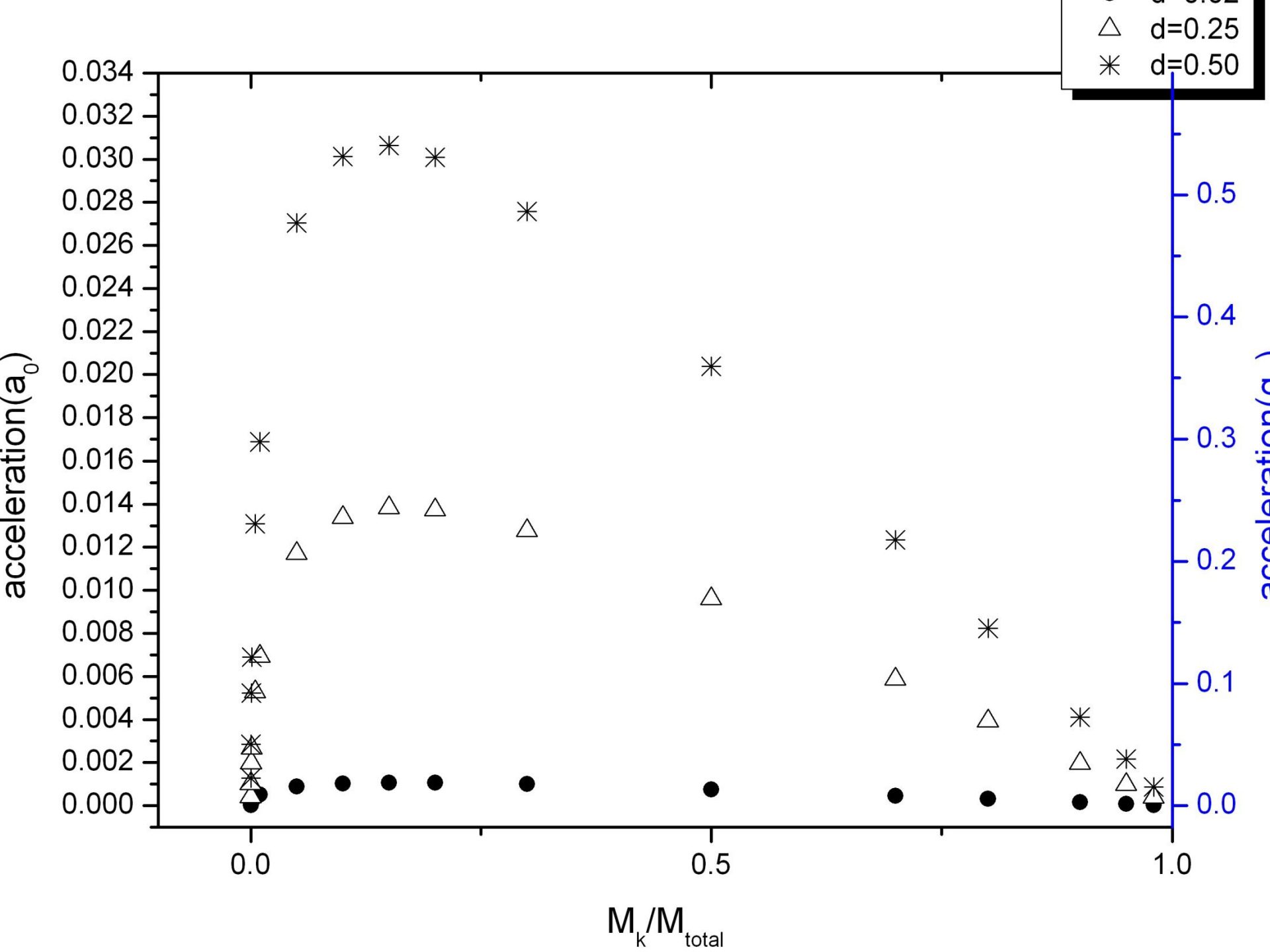
- The change due to the lattice size is less than .8%

- For the reference, we take a typical value of the MOND field just outside the mass shell to be

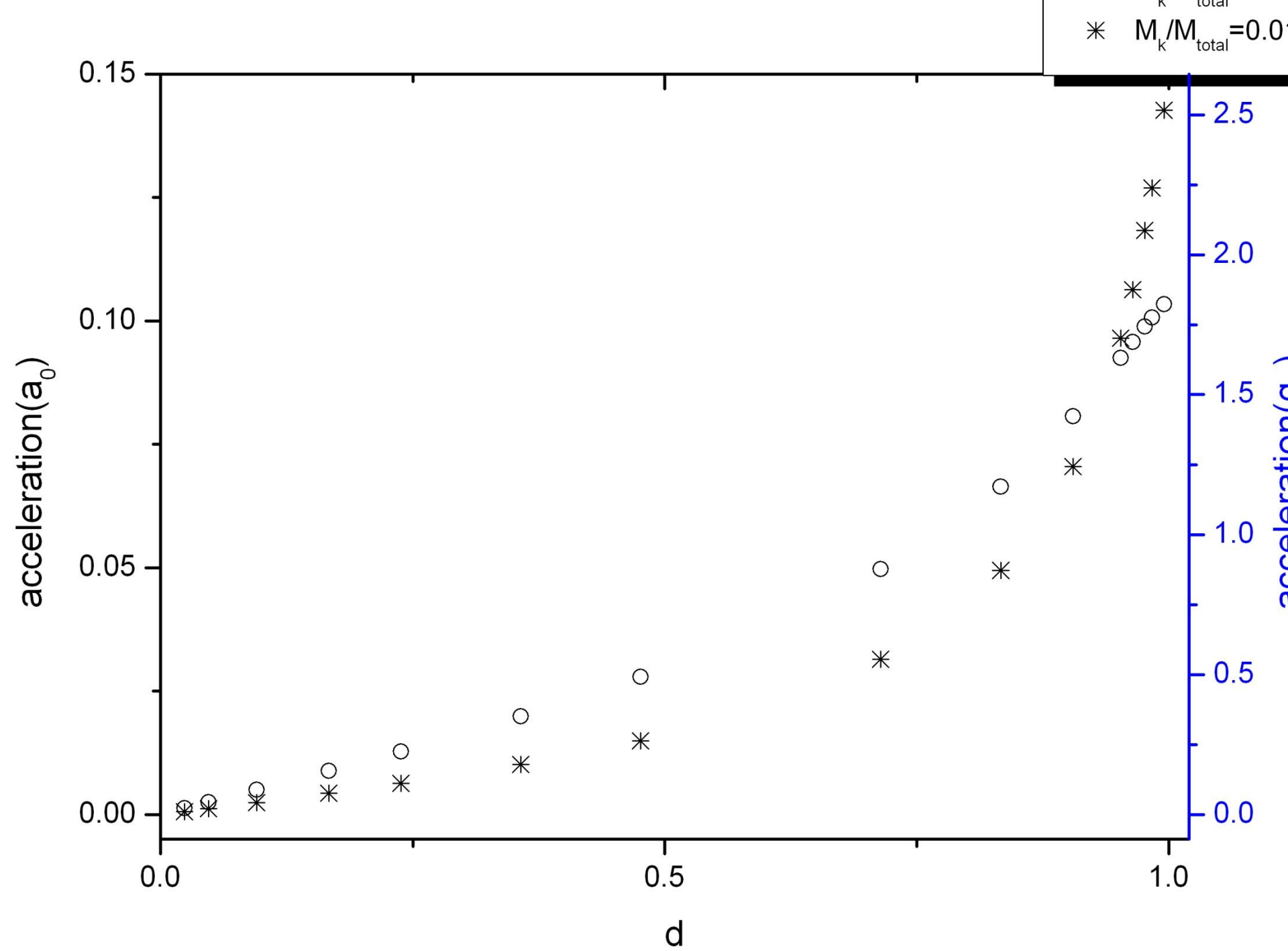
$$\frac{\sqrt{GM_{\text{total}}a_0}}{R_{\text{shell}}} = 0.24a_0$$

and a typical value of the Newtonian field to be

$$\frac{GM_{\text{total}}}{R_{\text{shell}}^2} = 0.057a_0$$







# Conclusion

- In the MOND theory, a test particle is accelerated in the spherical shell. The acceleration acts in the direction to restore the spherical symmetry (toward the center of the shell).
- The acceleration on the test particle can be a significant fraction of the typical acceleration just outside the mass shell.