Problem 1. (10 points) Variational principles are very powerful ways of solving problems. In class we have seen this in the context of Lagrangian mechanics (and will see it again in the context of Hamiltonian mechanics), but it is pervasive both within physics and without. Here we will consider a very different case that serves as the basis for numerical methods of solving boundary value problems. There is a theorem which states that (under certain conditions we will not worry about but assume they are all true) the unique solution for $y(x)$ to the differential equation

$$-\frac{d}{dx} \left[p(x) \frac{dy}{dx}\right] + q(x)y = f(x),$$

for $x \in [0, 1]$ is the function that minimizes the integral

$$I[u] = \int_0^1 \left\{ p(x)[u'(x)]^2 + q(x)[u(x)]^2 - 2f(x)u(x) \right\} dx.$$

Here the prime means derivative with respect to $x$. Differential equations of this form are very important in physics. It is written in self-adjoint form which immediately means that it involves a self-adjoint differential operator. In more common language, this means it involves a Hermitian operator. Starting from the integral (2) and the requirement that $\delta I = 0$ along the “true” path, $u(x) = y(x)$, clearly derive the differential equation (1). [Note: We will not show that this is a minimum, nor that it is unique. These can be done given the appropriate conditions on $p(x)$ and $q(x).$]

Problem 2. (15 points) A uniform, solid disk of mass $M$ and radius $R$ with a massless string wrapped around it is allowed to fall under gravity, as shown in the figure at the right. One end of the string is attached to a support at the top. The center of the disk falls in the vertical direction only (there is no motion to the side) as the string unwinds without stretching. Our goal is to find the acceleration of the center of the disk and the tension in the string. [Note: The point of this problem is the technique, not the answer. It is likely you have either seen this problem before, or have even solved it yourself. The “clearly” below means that the presentation of the technique used to solve the problem will be carefully scrutinized. The choices, assumptions, and steps must be clearly stated. Of course, the two methods also better give the same answer!]

(a) Clearly solve this problem using Newton’s second law.

(b) Clearly solve this problem using Lagrangian mechanics.

Problem 3. (20 points) Consider a particle of mass $m$ moving in the three-dimensional central potential

$$V(r) = -\frac{k}{r - R}.$$  

(a) This potential is very similar to a well known, well studied potential. State what potential this resembles and in what limit the given potential reduces to the well known one. [Note: Whenever we do a calculation we should have limiting or simple test cases in mind. We should always check our results at each step against these test cases to see if the results make
(b) Write down the Lagrangian for this system. Calculate the equations of motion. State what quantity (or quantities) is (or are) conserved, and why.

(c) Let $r_0$ be the radius of a circular orbit in this potential. Derive an algebraic relationship for $r_0$. You do not need to solve this equation.

(d) Consider a small perturbation of a circular orbit,

$$r(t) = r_0 + \delta r(t).$$

Derive the differential equation for $\delta r$ to lowest order. Write your differential equation in terms of $r_0$, $R$, $k$, $m$, and $\delta r$, only. Finally, use this equation to find the square of the frequency of the oscillations around the circular orbit.

(e) Clearly show that the circular orbit is stable in two limits, when $r_0 < n_1 R$ and when $r_0 > n_2 R$. Also find these two integers $n_1$ and $n_2$.

(f) Whenever we wish to study general properties of a system it is best to write it in terms of dimensionless quantities. In this case it is convenient to write $r_0$ in units of $R$ by defining $\zeta \equiv r_0/R$. Rewrite your algebraic expression for the radius in terms $\zeta$. You should find that the solution to this equation only depends on a dimensionless parameter,

$$\alpha \equiv \frac{\ell^2}{mkR},$$

where $\ell$ is the magnitude of the angular momentum.

(g) Graphically show that there is a critical value, $\alpha_c$, for which the system transitions from having a single circular orbit to having three circular orbits. Clearly state the number of circular orbits for $\alpha < \alpha_c$ and $\alpha > \alpha_c$.

(h) Analytically determine the numerical value for $\alpha_c$. It turns out to be a rational number!

**Problem 4.** (15 points) Here we will consider elastic scattering from an ellipsoid of revolution. If we choose the $z$-axis as the symmetry axis of the ellipsoid, then, in cylindrical coordinates we can write the ellipsoid as

$$\left( \frac{r}{r_0} \right)^2 + \left( \frac{z}{z_0} \right)^2 = 1,$$

where $r_0 = z_0 \sqrt{1 - e^2}$ and $e$ is the eccentricity of the ellipse in the $rz$-plane. Consider a beam of particles sent in parallel to the $z$-axis. Our objective to is to calculate the differential cross section, $\sigma(\theta)$.

(a) When I found this problem online, it claimed the differential cross section is given by

$$\sigma(\theta) = \frac{r_0^2 (1 - e^2)}{4 (1 - e^2 \cos^2(\theta/2))}.$$

It is hard to judge whether this is correct or not without repeating the calculation. They did the next logical thing and calculated the total cross section, finding

$$\sigma_T = \pi r_0^2 \left( \frac{1}{e^2} - 1 \right) \ln \left( \frac{1}{1 - e^2} \right).$$
They then proceeded to show in the limit of a sphere ($e \to 0$) this reduces to the known result from hard sphere scattering. All of this is great, but this total cross section is clearly wrong. It is good to test known limits, but that does not guarantee the answer is correct. In this case, geometrically we know without calculation what the total cross section must be. Write down the total cross section for hard ellipsoid scattering for the case we are considering and explain how you know it is correct.

(b) Correctly calculate the differential cross section for hard ellipsoid scattering. Verify that it produces the correct total cross section by directly integrating your expression for $\sigma(\theta)$. [Note: The differential cross section given above is not totally crazy. In the end there is just a simple typo in it. Even so, it is disturbing that it could be used to get a nonsensical answer and that such an answer would show up in a lengthy set of notes painstakingly typeset in \LaTeX. There must be some lessons we can learn from this about blindly trusting claims without thinking about them . . .]