1) (i) Starting from 
\[ y \frac{df}{dy} - y \frac{d}{dx} \left( \frac{df}{dy} \right) = 0 \]

we see that this is total. 

we note that since \( f(y, y') \) we have that 
\[
\frac{df}{dx} = \frac{df}{dy} \cdot y + \frac{df}{dy'} \cdot y' dx = \frac{df}{dy} \cdot y \frac{dy}{dx} = \frac{df}{dy} \cdot \frac{dy}{dx} \cdot \frac{dy'}{dy} \cdot \frac{df}{dy'} = \frac{df}{dy} \cdot \frac{dy}{dx} \frac{dy'}{dy} \frac{df}{dy'}.
\]

Thus 
\[
y \frac{df}{dy} - y \frac{d}{dx} \left( \frac{df}{dy} \right) = \frac{df}{dx} - \frac{df}{dy} \frac{dy}{dx} \frac{df}{dy'} = \frac{df}{dx} - \left( y \frac{df}{dy} \right) = \frac{df}{dx} - y \frac{df}{dy}.
\]

So the original equation is equivalent to 
\[
\frac{df}{dx} - y \frac{df}{dy} = c \Rightarrow f - y \frac{df}{dy} = c
\]

(ii) For the brachistochrone we had \( f(y, y') = \sqrt{1 + y'^2} \).

so \( \frac{df}{dy} = \frac{y}{\sqrt{y(1+y'^2)}} \).

From (i) we see that 
\[
\sqrt{\frac{1+y'^2}{y}} - \frac{y^2}{\sqrt{y(1+y'^2)}} = c \Rightarrow \frac{1}{\sqrt{y(1+y'^2)}} = c.
\]

\[
\Rightarrow y(1+y'^2) = \frac{1}{c^2} \Rightarrow \frac{y'^2}{c^2} = \frac{1}{c^2} - 1 \Rightarrow y' = \sqrt{\frac{1-c^2}{c^2}}.
\]

We can integrate this which gives 
\[
K = \int \sqrt{\frac{c^2y}{1-c^2y}} \, dy.
\]

To perform the integral let \( y = a \sin^2 \frac{\theta}{2} \) where \( a = \frac{1}{c^2} \)

then \( dy = a \sin \frac{\theta}{2} \cos \frac{\theta}{2} \, d\theta \).
With this we have

\[ x = \sqrt{\frac{a \sin^2 \theta}{a(1-\sin^2 \theta)}} \ a \sin \frac{x}{2} \cos \frac{x}{2} \ dy \]

\[ = \sqrt{\frac{\sin \theta}{\cos \theta}} \ a \sin \frac{x}{2} \cos \frac{x}{2} \ dy = \int \sin^2 \frac{x}{2} \ dy \]

\[ = b \quad (y - \sin y) + b \]

Choosing initial conditions \( x(0) = 0, y(0) = 0 \) implies \( y(0) = 0 \) mean that \( b = 0 \). Finally, absorbing the \( \frac{1}{2} \) into a
we arrive at \( x = a \left( y - \sin y \right) \)

We can rewrite \( y \) by noting that

\[ \cos y = 1 - 2 \sin^2 \frac{y}{2} \implies 2 \sin^2 \frac{y}{2} = 1 - \cos y. \]

Thus \( y = a \left( 1 - \cos y \right) \)

(iii) Consider \( L = T - V = \frac{1}{2} m q^2 - V(q) = L(q, q) \)

\[ \implies \frac{\partial L}{\partial q} = m\dot{q}. \quad \text{So from (i) we have} \]

\[ \frac{1}{2} m q^2 - V(q) - m q^2 = c \implies -\left[ \frac{1}{2} m \dot{q}^2 + V(q) \right] = c \]

\[ \implies T + V = \text{constant} \quad \text{so total energy is conserved.} \]

Notice that this is nothing more than \( h = q \frac{2L}{\partial q} - L \) being constant when \( L \) is independent of \( t \).
2. (i) The equation for a catenary is $x = a \cosh \left( \frac{y - b}{a} \right)$. 

When $x_1 = x_2$ and $y_1 = y_2$,

$\Rightarrow a \cosh \left( \frac{y_1 - b}{a} \right) = a \cosh \left( \frac{y_2 - b}{a} \right) \Rightarrow b = 0$

From $x_2 = a \cosh \left( \frac{y_2}{a} \right)$ let $k = \frac{y_2}{a}$, $\alpha = \frac{x_2}{y_2}$

$\Rightarrow \frac{x_2}{y_2} = \cosh \left( \frac{y_2}{a} \right) \Rightarrow k \alpha = \cosh (k)$

(ii) Graphically, a $\cosh$ looks much like a parabola. So we have the plot at the right, where $k \alpha$ is a line of slope $\alpha$.

We see there are 3 regions:

(1) $\alpha > \alpha_0$ : $k \alpha$ intersects the $\cosh$ at 2 points.

(2) $\alpha = \alpha_0$ : one solution

(3) $\alpha < \alpha_0$ : no solution

(iii) To find $\alpha_0$, we extremize $\frac{dx}{dk} = 0$.

So $k \alpha = \cosh (k) \Rightarrow \alpha + k \frac{d \alpha}{dk} = \sinh k$

$\Rightarrow \alpha = \sinh k$.

Plugging in we thus need to solve $k \sinh k = \cosh k$

$\Rightarrow k = \coth (k)$

Numerically $\Rightarrow k = 1.199679$

and $\alpha_0 = \sinh k = 1.508880$
3) \( T = \sum_i f_i (q_i) q_i^2, \quad V = \sum_i V_i (q_i) \)

(i) \( L = T - V = \sum \left[ f_i (q_i) \dot{q}_i^2 - V_i (q_i) \right] = \sum L_i (q_i, \dot{q}_i) \)

So the Lagrangian separates and the EoM must also.

Explicitly, the EoM become

\[
\frac{d}{dt} \left[ 2 f_i (q_i) \dot{q}_i \right] - \frac{\partial f_i}{\partial q_i} \ddot{q}_i + \frac{\partial V_i}{\partial q_i} = 0 \quad \text{for each } i.
\]

(ii) \( T = \frac{1}{2} q^2 \dot{q}^2 \), \( V = \frac{1}{2} k q^2 \) \( \Rightarrow \) \( L = \frac{1}{2} q^2 \dot{q}^2 - \frac{1}{2} k q^2 \)

EoM: \( \frac{d}{dt} (q^2 \dot{q}) - q \dddot{q} + k \dot{q} = 0 \)

\( \Rightarrow \) \( q \dddot{q} + q^2 \ddot{q} - q \dddot{q} + k \dot{q} = 0 \). Multiply by \( q \) to get \( q \dot{q}^3 + q^2 \ddot{q} + kq \dot{q} = 0 \), which is a total derivative

\( \Rightarrow \) \( \frac{1}{2} \frac{d}{dt} (q^2 \dot{q}^2) + \frac{1}{2} k \frac{d}{dt} q^2 = 0 \) \( \Rightarrow \) \( \frac{d}{dt} (q^2 \dot{q}^2 + kq^2) = 0 \)

\( \Rightarrow \) \( q^2 \dot{q}^2 + kq^2 = a \) where \( a \) is a constant

\( \Rightarrow \) \( \dot{q} = \sqrt{\frac{a - kq^2}{q^2}} \) \( \Rightarrow \) \( t = \int \sqrt{\frac{q^2}{a - kq^2}} \) dq

\( \Rightarrow \) \( t = -\sqrt{\frac{a - kq^2}{k}} + b \), inverting we have

\[
q(t) = \frac{a - k^2 (t - b)^2}{k}
\]

This can be plugged back into the EoM to verify it is correct!
For the general L from (i) the Eqn is
\[ \frac{d}{dt} \left( 2 \mathbf{f}_i \dot{\mathbf{q}}_i \right) - \frac{\partial \mathbf{f}_i}{\partial \mathbf{q}_i} \mathbf{q}_i \dot{\mathbf{q}}_i^2 + \frac{\partial \mathbf{V}_i}{\partial \mathbf{q}_i} = 0 \]

Since \( \mathbf{f}_i (\mathbf{q}_i) \) \( \Rightarrow \) \[ \frac{d \mathbf{f}_i}{dt} = \frac{\partial \mathbf{f}_i}{\partial \mathbf{q}_i} \mathbf{q}_i \]

\[ \Rightarrow \frac{d}{dt} \mathbf{f}_i \dot{\mathbf{q}}_i + 2 \mathbf{f}_i \dot{\mathbf{q}}_i \dot{\mathbf{q}}_i - \frac{\partial \mathbf{f}_i}{\partial \mathbf{q}_i} \dot{\mathbf{q}}_i^2 + \frac{\partial \mathbf{V}_i}{\partial \mathbf{q}_i} = 0 \]

Multiplying by \( \dot{\mathbf{q}}_i \) gives
\[ \frac{d}{dt} \mathbf{f}_i \dot{\mathbf{q}}_i^2 + 2 \mathbf{f}_i \mathbf{q}_i \dot{\mathbf{q}}_i \dot{\mathbf{q}}_i - \dot{\mathbf{q}}_i \frac{\partial \mathbf{V}_i}{\partial \mathbf{q}_i} = 0 \]

\[ \Rightarrow \frac{d}{dt} \left( \mathbf{f}_i \dot{\mathbf{q}}_i^2 \right) = - \frac{d \mathbf{V}_i}{dt} \]

\[ \Rightarrow \mathbf{f}_i \dot{\mathbf{q}}_i^2 = - \mathbf{V}_i + a \]

\[ \Rightarrow \dot{\mathbf{q}}_i^2 = \frac{a - \mathbf{V}_i}{\mathbf{f}_i} \]

\[ \Rightarrow t = \int \sqrt{\frac{\mathbf{f}_i (\mathbf{q}_i)}{a - \mathbf{V}_i (\mathbf{q}_i)}} \, d\mathbf{q}_i \]

Notice with \( \mathbf{f}_i (\mathbf{q}_i) = \mathbf{q}^2 \) this reproduces (ii).
i) Using the suggested coordinates

\[ T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \]

\[ V = \frac{1}{2} k (l - l_0)^2 \]

with the constraints \( x_1^2 + y_1^2 = a^2 \), \( x_2^2 + y_2^2 = b^2 \),
\[ l^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + c^2 \]

Thus we have

\[ L = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) - \frac{1}{2} k (l - l_0)^2 \]

\( f_1 = x_1^2 + y_1^2 - a^2 \), \( f_2 = x_2^2 + y_2^2 - b^2 \)
\[ f_3 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + c^2 - l^2 \]

ii) EoM:

\[ \begin{align*}
\ddot{x}_1: & \quad m_1 \ddot{x}_1 = -2 \lambda_1 x_1 + y_1, \\
\ddot{y}_1: & \quad m_1 \ddot{y}_1 = -2 \lambda_1 y_1 - 2 \lambda_3 (x_1 - x_2) \cup \text{constraints} \\
\ddot{x}_2: & \quad m_2 \ddot{x}_2 = -2 \lambda_2 x_2 + 2 \lambda_3 (x_1 - x_2) \\
\ddot{y}_2: & \quad m_2 \ddot{y}_2 = -2 \lambda_2 y_2 + 2 \lambda_3 (y_1 - y_2) \\
l: & \quad k (l - l_0) = 2 \lambda_3 l
\end{align*} \]
From the eqn for $I$, we see that $\lambda_3$ looks like the force from a spring (Hooke's law):

$$\lambda_3 \propto -2\lambda_1 (x_1 - x_2)$$

From $x_1 = \gamma_0 + \gamma_2$ we see that

$$m\ddot{x}_1 = -2\lambda_1 x_1 - 2\lambda_3 (x_1 - x_2), \quad \text{and similarly for } y_1.$$  

This is Newton's second law with 2 forces, the spring force and one that goes like $-2\lambda_1 x_1$ and $-2\lambda_1 y_1$.

This is a force $\bar{F} = -2\lambda_1 (x_1 \dot{x}_1 + y_1 \dot{y}_1)$.

This is exactly a "normal" or "tension" force that holds $M_1$ on the circle.

Thus $\lambda_1$ and $\lambda_2$ are the "normal" forces for these two particles.

iii) If $k=0$ then $\lambda_3 = 0$.

The constraint $x_1^2 + y_1^2 = a^2$ gives

$$\dot{x}_1 \dot{x}_1 + y_1 \dot{y}_1 = 0 \Rightarrow x_1 \dddot{x}_1 + x_1^2 + y_1 \dddot{y}_1 + y_1^2 = 0$$

$$\Rightarrow x_1 \dddot{x}_1 + y_1 \dddot{y}_1 = -\dot{x}_1^2 - \dot{y}_1^2 = -v_1^2.$$  

Combining $x_1 \dddot{x} + y_1 \dddot{y}$ gives

$$m_1 (x_1 \dddot{x}_1 + y_1 \dddot{y}_1) = -2\lambda_1 (x_1^2 + y_1^2)$$

$$\Rightarrow -m_1 v_1^2 = -2\lambda_1 a^2 \Rightarrow \lambda_1 = \frac{m_1 v_1^2}{2a^2}$$

Similarly, for $\lambda_2$:

$$\lambda_2 = \frac{m_2 v_2^2}{2b^2}$$

Plugging back in we find

$$\dddot{x}_1 = -\frac{v_1^2}{a} x_1, \quad \dddot{y}_1 = -\frac{v_1^2}{a} y_1$$

These are precisely Newton's second law for a particle moving in a circle.

The term $\frac{v^2}{a} (x_1 \dot{x}_1 + y_1 \dot{y}_1)$ is the centripetal acceleration.
(i) This is a central potential independent of time so we immediately know that angular momentum and energy are conserved. Thus the equation of motion can be written as first integrals:

\[
\begin{align*}
\dot{\theta} &= \frac{l}{m r^2} \\
\dot{r} &= \pm \sqrt{E^2 - \frac{l^2}{2m r^2} + \frac{k}{r} e^{-\alpha r}}
\end{align*}
\]

This behavior differs from gravity (for \(l \neq 0\)).

- For small \(r\): \(V_{eff} \sim \frac{l^2}{2mr^2}\) (like gravity)
- For large \(r\): \(V_{eff} \sim \frac{l^2}{2mr^2}\), since \(e^{-\alpha r} \to 0\), this is unlike gravity.

In both of these limits \(V_{eff} > 0\) so we will only have a minimum with \(V_{eff} < 0\) if \(l\) is not too large.

Thus we require \(\frac{l^2}{2mr^2} - \frac{k}{r} e^{-\alpha r} < 0\) \(\Rightarrow l^2 < 2mk e^{-\alpha r}\)

\[\Rightarrow l_{\text{max}} \approx \sqrt{2mk} e^{-\alpha r/2}\]

For \(l > l_{\text{max}}\) the Yukawa term never dominates so we do not have an extremum which means no circular orbit.
(i) \( \text{Let } \frac{dv}{dr} \bigg|_p = 0 \)

\[ 0 = -\frac{l^2}{mr^3} + \left( \frac{k}{r^2} + \frac{k}{ar^2} \right) e^{-r/a} \]

\[ \Rightarrow \frac{1}{m} \frac{d^2}{dr^2} = k \rho (1 + \frac{r}{a}) e^{-r/a} \Rightarrow \frac{l^2}{m ka} = \frac{p}{a} (1 + \frac{p}{a}) e^{-r/a}. \]

Let \( \alpha = \frac{l^2}{m ka}, \quad \chi = \frac{r}{a} \), then this equation is of the form

\[ \alpha = \chi (1 + \chi) e^{-\chi} \equiv g(\chi). \] Notice that \( \chi \) is a maximum when \( \chi \) is a maximum.

\[ \alpha_{\text{max}} \text{ occurs when } \frac{d\alpha}{d\chi} = 0 = \left[ 1 + \chi + \chi - \chi (1 + \chi) \right] e^{-\chi} \]

\[ \Rightarrow 1 + \chi - \chi^2 = 0 \Rightarrow \chi = 1 + \frac{\sqrt{5}}{2}. \] We require \( p > 0 \Rightarrow \chi > 0 \)

\[ \Rightarrow \chi = \frac{1 + \sqrt{5}}{2} = \phi \] the golden ratio!

Thus \( \alpha_{\text{max}} = g(\phi) \approx 0.84 \Rightarrow \alpha_{\text{max}} = \sqrt{g(\phi) m ka} \approx 0.92 m ka \)

For this orbit \( p = \phi a \approx 1.62 a \)

(iv) \( \text{We can proceed in a few ways. We could start from the EOM and perturb it or use one of our intermediate results from the general development. Here I start from } \)

\[ \beta^2 = 3 + \left( \frac{r}{f} \frac{df}{dr} \right) \bigg|_p. \]

For \( V = -\frac{k}{r} e^{-r/a} \Rightarrow f = -\frac{dV}{dr} = -k \left( \frac{1}{r^2} + \frac{1}{ar^2} \right) e^{-r/a} \]

\[ \Rightarrow \frac{df}{dr} = -k \left( -\frac{2}{r^3} - \frac{1}{ar^2} - \frac{1}{ar^2} - \frac{1}{a^2 r} \right) = k \left( \frac{2}{r^3} + \frac{2}{ar^2} + \frac{1}{a^2 r} \right) e^{-r/a}. \]
\[
\frac{\partial g}{\partial f} = - \frac{\rho}{1 + \alpha \rho} \left( 2 + \frac{2}{\alpha} \rho + \frac{1}{\alpha^2} \rho^2 \right) = - \frac{1}{1 + \alpha \rho} \left( 2 + \frac{2}{\alpha} \rho + \frac{1}{\alpha^2} \rho^2 \right).
\]

For \( \alpha \rho \ll 1 \) we expand to 2nd order in \( \alpha \rho \) (since the first order term cancels):

\[
\left( \frac{\partial g}{\partial f} \right) \approx \left( 1 - \frac{\rho}{\alpha^2} \right) \left( 2 + \frac{2}{\alpha} \rho + \frac{\rho^2}{\alpha^2} \right) - \frac{\rho^2}{\alpha^2} + \frac{1}{\alpha^3} \rho^3.
\]

Thus \( \beta^2 = 3 - 2 - \frac{\rho^2}{\alpha^2} = 1 - \frac{\rho^2}{\alpha^2} \).

So \( \beta = \sqrt{1 - \frac{\rho^2}{\alpha^2}} \approx 1 - \frac{1}{2} \frac{\rho^2}{\alpha^2} \).

After one orbit the particle will not return to the same point, instead:

\[
\Delta \theta = 2\pi (1 - \beta) = 2\pi \left( 1 - 1 + \frac{1}{2} \frac{\rho^2}{\alpha^2} \right).
\]

\[
\Delta \theta = \pi \frac{\rho^2}{\alpha^2}.
\]
\( \rho = \text{constant} \Rightarrow \rho = \frac{M(r)}{\frac{4}{3} \pi r^3} = M(r) = \frac{4}{3} \pi r^3 \rho. \)

(i) From Gauss' law, a planet of mass \( m \) at a distance \( r \) from the Sun will feel an additional gravitational force from the mass inside radius \( r \). Thus

\[
\vec{F}(r) = -\frac{GMmM(r)}{r^2} \hat{r} = -\frac{4}{3} \pi r^3 \rho \frac{GMm}{r^2} \hat{r}
\]

So

\[
\vec{F}(r) = -Cm \hat{r}
\]

where \( C = \frac{4}{3} \pi r^3 \rho G \).

(ii) Our equation of motion now becomes

\[
m \ddot{r} = \frac{\ell^2}{mr^3} - \frac{\hbar}{r^2} - mC \dot{r}.
\]

For a circular orbit, \( \ddot{r} = 0 \) at \( r = r_0 \)

\[
\Rightarrow \frac{d^2}{dr_0^2} = \frac{\hbar}{r_0^2} + mC r_0.
\]

So

\[
\ell = \sqrt{mk^2 r_0 + m^2 C r_0^4}.
\]

Angular momentum is still conserved (this is a central force) with

\[
\ell = mr_0^2 \Rightarrow \theta = \omega = \frac{\ell}{mr_0^2}.
\]

Thus

\[
W = \frac{\ell}{mr_0^2} = \frac{\hbar}{mr_0^2} \sqrt{mk^2 r_0 + m^2 C r_0^4} = \frac{\hbar}{mr_0^2} \sqrt{1 + \frac{mC}{k} r_0^3}
\]

\[
= \sqrt{\frac{\hbar}{mr_0^2}} \sqrt{1 + \frac{mC}{k} r_0^3}.
\]

The period is given by \( T = \frac{2\pi}{\omega} \). For gravity (Kepler's 3rd law)

Circular orbits have \( T_0 = 2\pi \sqrt{\frac{mr_0^3}{k}} \).

Thus

\[
T = T_0 \left(1 + \frac{mC}{k} r_0^3 \right)^{-1/2} \approx T_0 \left(1 - \frac{mC}{2k} r_0^3 \right).
\]
2) cont.

(iii) Let \( r = r_0 + Sr \). Again we can proceed in a number of ways. Here I will go back to the EqM. Plugging in this \( r \) we find after expanding

\[
M \frac{d^2 Sr}{dt^2} = \frac{l^2}{Mr^3} \left( \frac{1}{3} \frac{5 \frac{d^r}{r_0}}{r_0} \right) - \frac{k}{r_0^2} \left( \frac{1}{4} \frac{2 \frac{d^r}{r_0}}{r_0} \right) - mCr_0 \left( \frac{1}{4} \frac{8 \frac{d^r}{r_0}}{r_0} \right).
\]

leading order terms cancel for a circular orbit.

\[
\Rightarrow \frac{d^2 Sr}{dt^2} = \left( - \frac{3l^2}{Mr^3} + \frac{2k}{r_0^2} - Cr_0 \right) \frac{Sr}{r_0} = \frac{k}{Mr^3} \left( \frac{1}{4} + \frac{4mCr^3}{r_0} \right) \frac{Sr}{r_0}
\]

Using (ii) to simplify.

So we have a SHO with frequency

\[
\omega_r = \sqrt{\frac{k}{Mr^3}} \sqrt{1 + \frac{4mCr^3}{r_0}}, \quad \text{using } \tau_r = \frac{2\pi}{\omega_r} \text{ we have}
\]

\[
\tau_r = \tau_o \left( 1 + \frac{4mCr^3}{r_0} \right)^{-\frac{1}{4}} \approx \tau_o \left( 1 - \frac{2mCr^3}{r_0} \right)
\]

(iv) In (iii) we calculated \( \tau_o \), in (iii) we found \( \tau_r \). To leading order we see they are the same (as expected from Newtonian gravity).

To make them easier to manipulate notice we can rewrite them as

\[
\tau_o = \tau_o \left( 1 - \frac{C \tau_o^2}{8\pi^2} \right), \quad \tau_r = \tau_o \left( 1 - \frac{C \tau_o^2}{2\pi^2} \right).
\]

Since \( \tau_o > \tau_r \) the particle will travel from \( r = 0 \) back to \( r = 0 \) in less time than \( \tau_o \) goes from 0 to \( 2\pi \). Thus we should find the precession frequency is negative, \( \omega_p < 0 \).

We calculate this as

\[
\omega_p = \frac{2\pi}{\tau_o} - \frac{2\pi}{\tau_r} \approx 2\pi \tau_o \left[ 1 + \frac{C \tau_o^2}{8\pi^2} - 1 - \frac{C \tau_o^2}{2\pi^2} \right] = - \frac{2\pi}{\tau_o} \left( \frac{3C \tau_o^2}{8\pi^2} \right)
\]

\[
\Rightarrow \omega_p = - \frac{3C \tau_o}{8\pi} = - \frac{3C}{2} \frac{\sqrt{mr^3}}{r_0}
\]
\[ V(r) = -\frac{k}{r} + \frac{l}{r^2} \]

(i) We can proceed in a few ways. One is to note that we still have a central potential so \( l = m r^2 \dot{\phi} \) is still a conserved quantity.

From the Lagrange we can derive the GoM and will find

\[ \ddot{r} = -\frac{k}{r^2} + \frac{l^2 + 2m \hbar}{mr^3} \]

Alternatively we also know energy is conserved. This can be used to derive a first order differential equation:

\[ E = \frac{1}{2} m \dot{r}^2 + \frac{l^2}{2mr^2} - \frac{k}{r} = \frac{1}{2} m \dot{\phi}^2 - \frac{k}{r} + \frac{l^2 + 2m \hbar}{2mr^2} \]

In both cases we see the equations look exactly like gravity but with a modified angular momentum \( l' = l^2 + 2m \hbar \); this quantity is also conserved (since \( l \) is conserved and \( 2m \hbar \) is just a constant).

To connect this to our usual way of describing the motion let \( l' = m r^2 \dot{\phi} \) where we introduce a new angular quantity \( \phi \). It is related to the usual \( \Theta \) since

\[ \frac{l'}{l} = \frac{mr^2 \dot{\phi}}{mr^2 \dot{\Theta}} \Rightarrow \phi = \frac{l'}{l} \Theta \]

If we choose \( \phi(0) = \Theta(0) = 0 \) then we can write the orbit

\[ r(\phi) = \frac{l^{\prime 2}}{m \hbar} \left( \frac{1}{1 + e' \cos(\phi)} \right) \]

Notice the primes appearing. This is the same form as \( r(\Theta) \) but now using \( l' \) and \( \phi \).

Thus

\[ r(\phi) = \frac{l^{\prime 2}}{m \hbar} \left( \frac{1}{1 + e' \cos(\phi/2)} \right) \]

This describes a precessing ellipse if \( l' \neq 0 \).
(iii) Notice that \( l'^2 = l^2 + 2ml = \frac{l^2 + 2ml + m^2l^2}{l^2} \) 
\[ = l^2 \left(1 + \frac{2ml}{l^2} \right) = l^2 \left(1 + \frac{m}{l^2} \right). \]

If \( \frac{m}{l^2} \ll 1 \) then \( l' \approx l \sqrt{1 + 2m \frac{l}{l^2}} \approx l \left(1 + \frac{m}{l^2} \right) \).

Let \( l' = m \rho \dot{\Theta} + m \rho^2 \Omega \) for some new angular frequency \( \Omega \).

\[ l' = m \rho \dot{\Theta} \left(1 + \frac{\Omega}{\dot{\Theta}} \right) = l \left(1 + \frac{\Omega}{\dot{\Theta}} \right) \]

Comparing to the previous expression for \( l' \) we see

\[ \Omega = \frac{m \rho \dot{\Theta}}{l^2} = \frac{2\pi \rho \dot{\Theta}}{2\pi l^2} \quad \text{where} \quad \dot{\Theta} = \omega = \frac{2\pi}{T}. \]

(iii) The eccentricity is given by \( e = \sqrt{1 - \frac{c^2}{a^2}} \) so \( l'^2 = (1-e^2)ka \)

Thus \( \Omega = \frac{2\pi \rho \dot{\Theta}}{2\pi (1-e^2)ka} \Rightarrow \frac{\Omega}{\rho \dot{\Theta}a} = \frac{2\pi}{2\pi (1-e^2)} \quad \Omega = \eta \)

Using \( e = 0.2016, \quad \tau = 0.24 \text{yr}, \quad \Omega = 40'' \text{year}^{-1} \)

we find \( \eta = \frac{\Omega}{\dot{\Theta}a} = 7.1 \times 10^{-8} \)

(iv) Though the additional term has the same form as angular momentum, it is NOT the angular momentum that goes into determining the motion. In other words, the \( l \) of the system has not actually changed due to the perturbation and it is \( l \) that determines the motion.