Problem 1. (10 points) Consider an object moving in an elliptical orbit in a Newtonian gravitational field. [Note: This is problem 3.24 from Goldstein, Poole, and Safko, 3rd edition.]

(i) Show that the radial speed can be written as
\[ \dot{r} = \frac{\omega a}{r} \sqrt{a^2 e^2 - (a - r)^2}. \]

[Hint: If you find yourself lost in algebra you are probably approaching the problem in a difficult way. We want to derive this result, not just show it is true by working backwards, and the derivation can actually be quite short.]

(ii) Write \( r \) in terms of the eccentric anomaly, \( \psi \), and integrate the resulting differential equation to derive Kepler’s equation. [Note: This problem shows another way of writing elliptical orbits and shows another way of deriving Kepler’s equation. It is useful to have alternative ways of writing results as they can be useful for both analytic and numerical work.]

Problem 2. (30 points) We have seen that attractive forces inversely proportional to \( r^2 \) have an extra hidden symmetry. We discussed this symmetry in the context of the Laplace-Runge-Lenz vector which has been discovered, forgotten, rediscovered, ..., many times. In practice there are many ways to quantify this hidden symmetry all of which have undergone a similar history and still remain largely forgotten today. Recall that for a particle of mass \( m \) (strictly speaking we really mean the reduced mass of the system) moving in the potential \( V(r) = -k/r \) the Laplace-Runge-Lenz vector can be written as
\[ A = p \times L - mk\hat{r}. \]

In this problem we will study another conserved quantity, the Hamilton vector, written as
\[ u = v - \frac{k}{\ell} \hat{\theta}, \]
where \( v \) is the velocity of a particle in a plane, \( \ell \) is the magnitude of the angular momentum, and we are writing the system in polar coordinates \( (r, \theta) \). [Note: This problem looks very long since I have broken the calculations into many small steps.]

(i) Before focusing on the Kepler problem, let us consider a general central force \( F(r) = f(r)\hat{r} \). Calculate \( du/dt \) for this general force. Express your answer in terms of \( m, k, r, \ell, \) and \( f(r) \), only.

(ii) Use the result of the previous part to show that \( u \) is only conserved for an attractive force inversely proportional to \( r^2 \) (the Kepler problem).

(iii) Since gravity is super maximally integrable and we have previously found all the conserved quantities, the Hamilton vector cannot be new and must instead be related to other conserved quantities. Clearly show that
\[ A = mu \times L. \]

†The idea for this problem is based on http://arxiv.org/abs/0802.2431 though I have modified the notation. See that paper for other references to the history and usage of the Hamilton and related vectors.
(iv) Equivalently, clearly show that

\[ \mathbf{u} \propto \mathbf{L} \times \mathbf{A} \]

and find the proportionality constant.

(v) Finally, clearly show that

\[ u = |\mathbf{u}| = \frac{ke}{\ell}, \]

where \( e \) is the eccentricity of the orbit. [Hint: You know the magnitude of the Laplace-Runge-Lenz vector, \( |\mathbf{A}| \).]

We can use this new vector just as we used the Laplace-Runge-Lenz vector to algebraically solve problems. Here we will study the precession of orbits. Consider a central potential of the form

\[ V(r) = -\frac{k}{r} + U(r), \]

where \( U(r) \) is a general central potential.

(vi) The presence of \( U(r) \) means that \( \mathbf{u} \) is no longer conserved. Clearly show that

\[ \frac{d\mathbf{u}}{dt} = -\frac{1}{m} \frac{dU}{dr} \hat{r}. \]

(vii) The precession rate of the orbit can be calculated as

\[ \Omega = \frac{\mathbf{u} \times \dot{\mathbf{u}}}{u^2} \]

(you do not need to show this, it follows from the similar property of \( \mathbf{A} \)). Clearly show that

\[ \Omega = \frac{\ell^2}{mk^2e^2} \left( r\dot{\theta} - \frac{k}{\ell} \right) \frac{dU}{dr} \hat{z}. \]

(viii) The precession angle, \( \Delta \theta_p \), in one orbit can now be calculated as

\[ \Delta \theta_p = \int_0^T \Omega \, dt, \]

where \( T \) is the orbital period of the unperturbed orbit. If we consider small perturbations (this is the first place where we have invoked an approximation) then to first order in perturbation theory we can use the unperturbed orbit, \( r(\theta) \), to describe the motion. Clearly show that

\[ \Delta \theta_p = \frac{\ell^4}{m^2k^3e} \int_0^{2\pi} \frac{\cos \theta}{(1 + e \cos \theta)^2} \frac{dU}{dr} \, d\theta. \]

[Note: This approach is standard in perturbation theory. It is probably most familiar to us in the calculation of the first order correction to the energy in quantum mechanics where we calculate the expectation value of the perturbing Hamiltonian using the unperturbed wave functions.]

(ix) Now that we have a new method for calculating precession we should apply it to a known result. In the previous homework we calculated the precession in the Yukawa potential,

\[ V(r) = -\frac{k}{r} e^{-r/a}. \]
Unfortunately we cannot directly use the formalism we just painstakingly developed with this potential. Explain why not.

(x) In the actual problem we solved we considered the case where \( r/a \ll 1 \). Expand \( V(r) \) to second order and determine \( U(r) \). \([Note: As we saw when we solved this problem in the homework, it was necessary to expand to second order since the first order terms “magically” canceled. When we use \( U(r) \) in the next part it should be “obvious” that the first order term does not contribute.]

(xi) Finally, calculate \( \Delta \theta_p \) for small perturbations from a circular orbit of radius \( \rho \). Confirm that you reproduce the result we found in the homework. \([Note: You may use some tool to perform the required integral. This problem has been long enough as it is.]

Problem 3. (10 points) Show that the differential cross section for scattering by a repulsive central force with magnitude \( f = k/r^3 \) is given by

\[
\sigma(\theta) \, d\theta = \frac{k}{2E} \frac{(1-x)dx}{x^2(2-x)^2 \sin(\pi x)},
\]

where \( x \equiv \theta/\pi \) and \( E \) is the energy.

Problem 4. (20 points) In nuclear physics, in particular, it is common to treat scattering of particles from nuclei as an optical problem. This “optical model” provides one way of understanding particle-wave duality by associating scattering with refraction/diffraction. Here we will study this connection through scattering from a spherical well,

\[
V(r) = \begin{cases} 
0 & r > a \\
-V_0 & r \leq a
\end{cases}.
\]

(i) Show that the scattering from this potential is equivalent to the refraction of light rays passing through a sphere of radius \( a \) and with a relative index of refraction

\[
n = \sqrt{\frac{E + V_0}{E}}.
\]

(ii) Show that the differential cross section for the scattering is

\[
\sigma(\theta) = \frac{n^2 a^2}{4 \cos(\theta/2)} \frac{[n \cos(\theta/2) - 1][n - \cos(\theta/2)]}{[1 + n^2 - 2n \cos(\theta/2)]^2}.
\]

(iii) Show that there is a maximum scattering angle and calculate its value.

(iv) From the optical analogue in part (i) we know what the total cross section must be. What is the total cross section? Also explicitly calculate the total cross section and verify that you get the known answer. \([Hint: You do not need an expensive, proprietary program to evaluate this integral. In fact you can easily perform it yourself and can even find a general expression for how we calculate the total cross section in terms of the impact parameter. Further, since we know what the answer must be and that it is simple expression, we should find a simple way to derive it.]

Problem 5. (10 points) A particle of mass \( m_1 \) with initial velocity \( v_1 \) elastically scatters from a particle of mass \( m_2 \), initially at rest.
(i) Calculate the surface of constant time travel for the scattered particle as a function of scattering angle. That is, find the surface as measured from the collision point that the particle of mass \( m_1 \) reaches at some fixed time, \( t_1 \), as a function of the scattering angle. As is often the case it is best to calculate a dimensionless quantity that can then be scaled depending on the details of the problem we are solving. In this case it is most convenient to calculate \( d_1(\theta_1) / v_1 t_1 \).

(ii) Consider your result from the previous part for the three cases \( m_2 \to \infty \), \( m_2 = 2m_1 \), and \( m_2 = m_1 \). You should think about what you expect the results to be before you calculate them. Calculate the surface in these three cases and plot or sketch it for these cases. Explain why they make sense. In particular, explain why your results for the two extreme cases, \( m_2 \to \infty \) and \( m_2 = m_1 \) are reasonable.

Problem 6. (30 points) Here we will study the transformation of cross sections between frames. Whenever we encounter something new like this it is best to carefully work through a simple example where we know the answers. Here the simple example we will consider is hard sphere scattering. We will see the benefit of working in the correct reference frame. Consider a particle of mass \( m \) which elastically scatters from a sphere of mass \( M \) and radius \( a \). [Note: Hard sphere scattering is just elastic scattering with the incident angle equal to the reflection angle.]

(i) Calculate the differential cross section, \( \sigma(\theta) \), and from this the total cross section, \( \sigma_T \). You should get the expected answer. This calculation is done in the center of mass frame.

(ii) In class we saw that

\[
\sigma_{\text{lab}}(\theta_{\text{lab}}) = \sigma_{\text{cm}}(\theta_{\text{cm}}) \left( \frac{\rho \cos \theta_{\text{lab}} + \sqrt{1 - \rho^2 \sin^2 \theta_{\text{lab}}}}{\sqrt{1 - \rho^2 \sin^2 \theta_{\text{lab}}}} \right)^2.
\]

(1)

For the case when \( M = m \) calculate \( \sigma_{\text{lab}}(\theta_{\text{lab}}) \) and integrate it to find \( \sigma_T \). The answer better agree with the previous part! [Hint: What is the maximum scattering angle?]

(iii) For the case when \( m < M \) again integrate (1) to find the total cross section. We know what the value of the integral must be since our answer must agree with the previous part. We want to verify that the value of the integral does come out correctly.

(iv) To transform from the center of mass frame to the lab frame we saw that

\[
\tan \theta_{\text{lab}} = \frac{\sin \theta_{\text{cm}}}{\rho + \cos \theta_{\text{cm}}}.
\]

Use this to show that

\[
\cos \theta_{\text{cm}} = -\rho \sin^2 \theta_{\text{lab}} \pm \cos \theta_{\text{lab}} \sqrt{1 - \rho^2 \sin^2 \theta_{\text{lab}}},
\]

(2)

[Note: This derivation and result is very similar to the spontaneous decay case.]

(v) We can now directly calculate the scattering cross section in the lab frame. Previously we calculated \( b(\theta_{\text{cm}}) \). Transform that expression using (2) to find \( \sigma_{\text{lab}}(\theta_{\text{lab}}) \), again for the case \( m < M \). The result should agree with the expression in part (ii). [Note: We do not need to perform endless pages of algebra. Recall the we are are calculating

\[
\sigma_{\text{lab}}(\theta_{\text{lab}}) = \frac{b(\theta_{\text{lab}})}{\sin \theta_{\text{lab}}} \left| \frac{db}{d\theta_{\text{lab}}} \right|.
\]

With this in mind we can organize our calculation in a smart way to greatly simplify the required algebra. We should always think about smart ways to organize calculations instead of entirely relying on the brute force approach.]
(vi) It is now time to deal with the case \( m > M \) which we have been avoiding. In this case there are two center of mass angles that map to the same lab angle. Again calculate \( \sigma_{\text{lab}}(\theta_{\text{lab}}) \) and integrate this expression to once again get the expected answer. [Note: The maximum scattering angle in the lab frame is not \( \pi \).]

**Problem 7.** (10 points) Suppose the the matrix

\[
R = \begin{pmatrix}
R_{00} & R_{01} & R_{02} \\
R_{10} & R_{11} & R_{12} \\
R_{20} & R_{21} & R_{22}
\end{pmatrix}
\]

is a rotation matrix calculated from Euler angles in the \( ZXZ \) scheme.

(i) We would like to extract the Euler angles, \((\psi, \theta, \varphi)\), from the matrix \( R \). Write down expressions for finding these angles in terms of the elements of \( R \).

(ii) Consider the specific matrix

\[
R = \begin{pmatrix}
0.37810741 & -0.83186439 & 0.40624675 \\
0.88151370 & 0.45756292 & 0.11648938 \\
-0.28278681 & 0.31406657 & 0.90630779
\end{pmatrix}.
\]

Apply your expressions from the previous part and find the Euler angles for this matrix. [Note: The angles are integers when written in degrees.]

(iii) Not all the elements of \( R \) are required to calculate the Euler angles. Using the angles you found in the previous part calculate the values you expect for the rest of the elements in \( R \). Do they match the given matrix? [Note: If you are not familiar with numerical work, particularly when dealing with inverse trigonometric functions, then I am expecting that not all the entries will agree. This is fine and even preferred.]

(iv) A difficulty of working with trigonometric functions is that they are multiple valued; for example,

\[
cos \frac{\pi}{3} = \cos \frac{7\pi}{3} = \frac{1}{2}.
\]

If we are given \( \cos \varphi = 1/2 \) how do we know which angle to choose? Since \( \varphi \in [0, 2\pi] \) we do not know! For \( \theta \in [0, \pi] \) we do not encounter this issue but for \( \varphi \) and \( \psi \) we do. Numerically the way to handle this is to use the arctangent whenever we can. Every useful numerical library/utility will provide a two argument version of the arctangent typically called something like \texttt{atan2} or \texttt{arctan2}. Assuming you did not do so from the beginning (don’t cheat, read this, and then use this information in the first part of the problem!) rewrite your expressions from the first part using the arctangent where you can, making sure you keep any signs with the appropriate part of the expression (in the numerator or denominator). Then use the two argument version of the arctangent to calculate the angles. You have now encountered one of the many joys of computational physics. [Hint: One of the angles was chosen to be larger than \( \pi \) to highlight this issue.]