Phys 413 Homework 1

Due: 10 September 2018

Problem 1. (10 points) An old hiking adage states

Never step on anything you can step over, and
never step over anything you can step around.

(i) In the wonderful world of introductory physics we do not need to follow this adage. Briefly explain why not. In fact, in such a wonderful world what determines whether you can complete the hike between two points? [Hint: What do we typically ignore in introductory physics?]

(ii) In the real world this adage does actually make sense and is good to follow. Based on simple physics, briefly explain why.†

Problem 2. (10 points) As we learn new, more powerful techniques and reformulate mechanics in abstract ways, we should not forget good old Newtonian mechanics. It is still the starting point for everything we do and provides much of our intuition of physical systems. Throughout the semester we will periodically go back to Newtonian mechanics. This problem is one example. Consider a metal block of mass $m$ sliding in one dimension on a horizontal surface that has been lubricated with a heavy oil. The block suffers a viscous resistive force

$$F(v) = -cv^{3/2},$$

for a constant, $c$. The block starts with an initial speed of $v_0$ at $x = 0$. Here we wish to calculate the maximum distance the block will travel before stopping.

(i) Our first thought may be to solve for $v(t)$ and $x(t)$. Do so and state the time, $\tilde{t}$, at which the block stops. Finally find the maximum distance traveled.

(ii) Since we are only interested in the maximum distance traveled we do not need to find the time, $\tilde{t}$. Instead, starting from the resistive force calculate $v(x)$ or $x(v)$ (whichever is more convenient). Use this to confirm your result from the previous part. [Note: Do not use your result from the previous part, directly perform the calculation starting from the force.]

Problem 3. (10 points) In chapter 1, derivation 4 of Goldstein, Poole, and Safko (GPS) the special case of linear differential constraints of the form

$$\sum_{j=1}^{n} g_j(x_1, \ldots, x_n) dx_j = 0$$

is considered.

(i) It is stated that this constraint condition is holonomic if an integrating function, $f(x_1, \ldots, x_n)$, can be found that makes this an exact differential. Show that this statement is true. [Note: What is an integrating function? This is often a point of confusion.]

(ii) It is further stated that if the integrating function, $f$, exists it must satisfy the relations

$$\frac{\partial (fg_i)}{\partial x_j} = \frac{\partial (fg_j)}{\partial x_i}, \quad \text{for all } i \neq j.$$
Show that this statement is true.

(iii) For the case of the rolling disk [see GPS Eq. (1.39) if you want more context] one of the constraint equations is

\[ dx - a \sin \theta \, d\varphi = 0. \]

Show that we cannot find an integrating function, \( f \), for this constraint thus showing that it is not holonomic.

**Problem 4.** (10 points)

(i) Show that the Lagrangian, \( L(q, \dot{q}, t) \), is invariant under the transformation

\[ L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d\Lambda(q, t)}{dt}, \]

that is, show by direct substitution that both \( L \) and \( L' \) have the same equations of motion. \([Note: \ This \ shows \ that \ the \ Lagrangian \ is \ not \ unique. \ We \ can \ always \ add \ a \ total \ time \ derivative \ (only \ depending \ on \ the \ generalized \ coordinates \ and \ time) \ and \ not \ change \ the \ physics.]

(ii) The set of generalized coordinates \( \{q_k\} \) is not unique. Transformations that only depend on another set of coordinates \( \{s_k\} \) (but not their time derivatives) where

\[ q_k = q_k(s_1, \ldots, s_n, t), \quad \text{for } k = 1, \ldots, n, \]

are called point transformations. Show that the Lagrangian is invariant under point transformations, that is, show that the equations of motion for \( L(s_k, \dot{s}_k, t) \) are

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{s}_k} \right) - \frac{\partial L}{\partial s_k} = 0. \]

**Problem 5.** (10 points) A particle of mass \( m \) moves in a smooth, one dimensional potential, \( V(x) \), such that it has the Lagrangian

\[ L = \frac{m^2 \dot{x}^4}{12} + m\dot{x}^2 V(x) - V^2(x). \]

Find the equation of motion for \( x(t) \) and describe the physical nature of the system on the basis of this equation.

**Problem 6.** (10 points) Two points of mass \( m \) are joined by a massless rod of length \( \ell \), the center of which is constrained to move in a circle of radius \( a \).

(i) Assuming no friction, describe the motion of the system. In other words, describe how \( \theta \) and \( \varphi \) change with time. Do not solve any equations; physically we know what the motion should be so just describe it. \([Note: \ Yes, \ this \ is \ suppose \ to \ be \ straightforward. \ It \ is \ to \ remind \ us \ that \ we \ should \ think \ about \ problems \ before \ solving \ them. \ It \ is \ also \ a \ less \ trivial \ example \ of \ how \ constraint \ forces \ can \ lead \ to \ the \ simple \ motion \ of \ a \ system \ and \ that \ they \ can \ silently \ be \ absorbed \ into \ our \ generalized \ coordinates \ without \ worrying \ about \ their \ details.]

(ii) Use the quantities in the figure at the right to express the kinetic energy of the system.
Since there is no potential energy the Lagrangian is given by the kinetic energy. Write down the Euler-Lagrange equations and show that they (trivially) reproduce the motion you described in the first part of this problem.

**Problem 7.** (10 points) Goldstein, Poole, and Safko claim that the integration of the brachistochrone problem is “straightforward”. This is true for a some definition of straightforward. Here we will explore the problem in a more general framework. Consider the case where $f$ does not explicitly depend on $x$, that is, $f(y, \dot{y})$ where $\dot{y} \equiv dy/dx$.

(i) If $\dot{y} \neq 0$ then the Euler-Lagrange equations may be written as

$$\dot{y} \frac{\partial f}{\partial y} - \dot{y} \frac{d}{dx} \left( \frac{\partial f}{\partial \dot{y}} \right) = 0.$$  

Starting from this form show that

$$f - \dot{y} \frac{\partial f}{\partial \dot{y}} = c$$

for $c$ an arbitrary constant. This result has many uses, including in some of the following problems.

(ii) Apply the previous result to the brachistochrone problem. Choose $(x_1, y_1) = (0, 0)$ and derive the parametric solutions,

$$x = a(\varphi - \sin \varphi), \quad y = a(1 - \cos \varphi),$$

where $a$ is a constant related to $c$. This is what they mean by “straightforward”. [*Hint: To perform the integration yourself make the trigonometric substitution $y = a \sin^2(\varphi/2)$.]*

(iii) Apply the result from part (i) to the Lagrangian for a particle of mass $m$ in the special case where $V(q)$ only. What quantity is found to be conserved? [*Note: This is a special case of a more general result.*]

**Problem 8.** (10 points) As stated in class, calculus of variations does not necessarily provide a minimum as we will now see in the case of the surface of minimum revolution. We saw in class that the extremum for the area of a surface of revolution is given by

$$x = a \cosh \left( \frac{y - b}{a} \right).$$

Consider the symmetric case, $x_2 = x_1$ and $y_2 = -y_1 > 0$ of the surface of revolution.

(i) For the symmetric case find $b$. Also, let

$$k \equiv \frac{y_2}{a} \quad \text{and} \quad \alpha \equiv \frac{x_2}{y_2}.$$

Find the transcendental equation relating $k$ and $\alpha$, that is, find an equation for $\alpha(k)$.

(ii) Show that there are three regions of solutions for $\alpha(k)$: (1) For $\alpha > \alpha_0$ two values of $k$ satisfy the equation, (2) at the particular minimum value, $\alpha_0$, only one value of $k$ satisfies the equation, and (3) for $\alpha < \alpha_0$ there are no solutions to the equation. This is most easily done graphically.

(iii) Numerically find the value of $\alpha_0$. 

3