PHYS 331 Homework 12 Solutions

9 December 2011

Problem 1. (10 points)

Problem 5.2

(a) From Eq. 4.77, E_1 is proportional to mass, so $\frac{\Delta E_1}{E_1} = \frac{\Delta m}{\mu} = \frac{m-\mu}{\mu} = \frac{m(m+M)}{mM} - \frac{M}{M} = \frac{m}{M}$.
The fractional error is the ratio of the electron mass to the proton mass:
$\frac{9.109 \times 10^{-31}\,\text{kg}}{1.673 \times 10^{-27}\,\text{kg}} = 5.44 \times 10^{-4}$. The *percent* error is $\boxed{0.054\%}$ (pretty small).

(b) From Eq. 4.94, R is proportional to m, so $\frac{\Delta(1/\lambda)}{(1/\lambda)} = \frac{\Delta R}{R} = \frac{\Delta \mu}{\mu} = -\frac{(1/\lambda^2)\Delta\lambda}{(1/\lambda)} = -\frac{\Delta\lambda}{\lambda}$.
So (in magnitude) $\Delta\lambda/\lambda = \Delta\mu/\mu$. But $\mu = mM/(m+M)$, where m = electron mass, and M = nuclear mass.

$$\Delta\mu = \frac{m(2m_p)}{m+2m_p} - \frac{mm_p}{m+m_p} = \frac{mm_p}{(m+m_p)(m+2m_p)}(2m + 2m_p - m - 2m_p)$$

$$= \frac{m^2 m_p}{(m+m_p)(m+2m_p)} = \frac{m\mu}{m+2m_p}.$$

$$\frac{\Delta\lambda}{\lambda} = \frac{\Delta\mu}{\mu} = \frac{m}{m+2m_p} \approx \frac{m}{2m_p}, \text{ so } \boxed{\Delta\lambda = \frac{m}{2m_p}\lambda_h}, \text{ where } \lambda_h \text{ is the hydrogen wavelength.}$$

$$\frac{1}{\lambda} = R\left(\frac{1}{4} - \frac{1}{9}\right)\frac{5}{36}R \Rightarrow \lambda = \frac{36}{5R} = \frac{36}{5(1.097 \times 10^7)}\,\text{m} = 6.563 \times 10^{-7}\,\text{m}.$$

$$\therefore \Delta\lambda = \frac{9.109 \times 10^{-31}}{2(1.673 \times 10^{-27})}(6.563 \times 10^{-7})\text{m} = \boxed{1.79 \times 10^{-10}\,\text{m.}}$$

(c) $\mu = \frac{mm}{m+m} = \frac{m}{2}$, so the energy is *half* what it would be for hydrogen: $(13.6/2)\text{eV} = \boxed{6.8\,\text{eV.}}$

(d) $\mu = \frac{m_p m_\mu}{m_p + m_\mu}$; $R \propto \mu$, so R is changed by a factor $\frac{m_p m_\mu}{m_p + m_\mu} \cdot \frac{m_p + m_e}{m_p m_e} = \frac{m_\mu(m_p + m_e)}{m_e(m_p + m_\mu)}$, as compared with hydrogen. For hydrogen, $1/\lambda = R(1 - 1/4) = \frac{3}{4}R \Rightarrow \lambda = 4/3R = 4/3(1.097 \times 10^7)\,\text{m} = 1.215 \times 10^{-7}\,\text{m}$, and $\lambda \propto 1/R$, so for muonic hydrogen the Lyman-alpha line is at

$$\lambda = \frac{m_e(m_p + m_\mu)}{m_\mu(m_p + m_e)}(1.215 \times 10^{-7}\text{m}) = \frac{1}{206.77}\frac{(1.673 \times 10^{-27} + 206.77 \times 9.109 \times 10^{-31})}{(1.673 \times 10^{-27} + 9.109 \times 10^{-31})}(1.215 \times 10^{-7}\text{m})$$
$$= \boxed{6.54 \times 10^{-10}\,\text{m.}}$$

Problem 2. (10 points)

Problem 5.5

(a)

$$\boxed{-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x_1^2}-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x_2^2}=E\psi} \quad \text{(for } 0 \le x_1, x_2 \le a, \text{ otherwise } \psi = 0\text{).}$$

$$\psi = \frac{\sqrt{2}}{a}\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right)-\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right]$$

$$\frac{d^2\psi}{dx_1^2}=\frac{\sqrt{2}}{a}\left[-\left(\frac{\pi}{a}\right)^2\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right)+\left(\frac{2\pi}{a}\right)^2\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right]$$

$$\frac{d^2\psi}{dx_2^2}=\frac{\sqrt{2}}{a}\left[-\left(\frac{2\pi}{a}\right)^2\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right)+\left(\frac{\pi}{a}\right)^2\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right]$$

$$\left(\frac{d^2\psi}{dx_1^2}+\frac{d^2\psi}{dx_2^2}\right)=-\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{2\pi}{a}\right)^2\right]\psi=-5\frac{\pi^2}{a^2}\psi,$$

$$-\frac{\hbar^2}{2m}\left(\frac{d^2\psi}{dx_1^2}+\frac{d^2\psi}{dx_2^2}\right)=\frac{5\pi^2\hbar^2}{2ma^2}\psi=E\psi, \quad \text{with} \quad E=\frac{5\pi^2\hbar^2}{2ma^2}=5K. \quad \checkmark$$

(b) Distinguishable:

$$\boxed{\psi_{22}=(2/a)\sin(2\pi x_1/a)\sin(2\pi x_2/a)\,, \text{ with } E_{22}=8K} \quad \text{(nondegenerate).}$$

$$\boxed{\left.\begin{array}{l}\psi_{13}=(2/a)\sin(\pi x_1/a)\sin(3\pi x_2/a)\\ \psi_{31}=(2/a)\sin(3\pi x_1/a)\sin(\pi x_2/a)\end{array}\right\}, \text{ with } E_{13}=E_{31}=10K} \quad \text{(doubly degenerate).}$$

Identical Bosons:

$$\boxed{\psi_{22}=(2/a)\sin(2\pi x_1/a)\sin(2\pi x_2/a),\ E_{22}=8K} \quad \text{(nondegenerate).}$$

$$\boxed{\psi_{13}=(\sqrt{2}/a)\left[\sin(\pi x_1/a)\sin(3\pi x_2/a)+\sin(3\pi x_1/a)\sin(\pi x_2/a)\right],\ E_{13}=10K} \quad \text{(nondegenerate).}$$

Identical Fermions:

$$\boxed{\psi_{13}=(\sqrt{2}/a)\left[\sin\left(\tfrac{\pi x_1}{a}\right)\sin\left(\tfrac{3\pi x_2}{a}\right)-\sin\left(\tfrac{3\pi x_1}{a}\right)\sin\left(\tfrac{\pi x_2}{a}\right)\right],\ E_{13}=10K} \quad \text{(nondegenerate).}$$

$$\boxed{\psi_{23}=(\sqrt{2}/a)\left[\sin\left(\tfrac{2\pi x_1}{a}\right)\sin\left(\tfrac{3\pi x_2}{a}\right)-\sin\left(\tfrac{3\pi x_1}{a}\right)\sin\left(\tfrac{2\pi x_2}{a}\right)\right],\ E_{23}=13K} \quad \text{(nondegenerate).}$$

Problem 3. (10 points)

Problem 5.6

(a) Use Eq. 5.19 and Problem 2.4, with $\langle x\rangle_n = a/2$ and $\langle x^2\rangle_n = a^2\left(\frac{1}{3} - \frac{1}{2(n\pi)^2}\right)$.

$$\langle(x_1 - x_2)^2\rangle = a^2\left(\frac{1}{3} - \frac{1}{2(n\pi)^2}\right) + a^2\left(\frac{1}{3} - \frac{1}{2(m\pi)^2}\right) - 2\cdot\frac{a}{2}\cdot\frac{a}{2} = \boxed{a^2\left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{n^2} + \frac{1}{m^2}\right)\right]}.$$

(b) $\langle x\rangle_{mn} = \frac{2}{a}\int_0^a x\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x\right)dx = \frac{1}{a}\int_0^a x\left[\cos\left(\frac{(m-n)\pi}{a}x\right) - \cos\left(\frac{(m+n)\pi}{a}x\right)\right]dx$

$$= \frac{1}{a}\left[\left(\frac{a}{(m-n)\pi}\right)^2\cos\left(\frac{(m-n)\pi}{a}x\right) + \left(\frac{ax}{(m-n)\pi}\right)\sin\left(\frac{(m-n)\pi}{a}x\right)\right.$$
$$\left.\left. - \left(\frac{a}{(m+n)\pi}\right)^2\cos\left(\frac{(m+n)\pi}{a}x\right) - \left(\frac{ax}{(m+n)\pi}\right)\sin\left(\frac{(m+n)\pi}{a}x\right)\right]\right|_0^a$$
$$= \frac{1}{a}\left[\left(\frac{a}{(m-n)\pi}\right)^2(\cos[(m-n)\pi] - 1) - \left(\frac{a}{(m+n)\pi}\right)^2(\cos[(m+n)\pi] - 1)\right].$$

But $\cos[(m\pm n)\pi] = (-1)^{m+n}$, so

$$\langle x\rangle_{mn} = \frac{a}{\pi^2}\left[(-1)^{m+n} - 1\right]\left(\frac{1}{(m-n)^2} - \frac{1}{(m+n)^2}\right) = \begin{cases} \frac{a(-8mn)}{\pi^2(m^2-n^2)^2}, & \text{if } m \text{ and } n \text{ have opposite parity,} \\ 0, & \text{if } m \text{ and } n \text{ have same parity.}\end{cases}$$

So Eq. 5.21 $\Rightarrow \langle(x_1 - x_2)^2\rangle = \boxed{a^2\left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{n^2} + \frac{1}{m^2}\right)\right] - \frac{128a^2m^2n^2}{\pi^4(m^2-n^2)^4}.}$

(The last term is present only when m, n have opposite parity.)

(c) Here Eq. 5.21 $\Rightarrow \langle(x_1 - x_2)^2\rangle = \boxed{a^2\left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{n^2} + \frac{1}{m^2}\right)\right] + \frac{128a^2m^2n^2}{\pi^4(m^2-n^2)^4}.}$

(Again, the last term is present only when m, n have opposite parity.)

Problem 4. (5 points)

Problem 5.33

(a) Each particle has 3 possible states: $3\times3\times3 = \boxed{27.}$

(b) All in same state: aaa, bbb, $ccc \Rightarrow 3$.

2 in one state: aab, aac, bba, bbc, cca, $ccb \Rightarrow 6$ (each symmetrized).

3 different states: abc (symmetrized) $\Rightarrow 1$.

Total: $\boxed{10.}$

(c) Only abc (antisymmetrized) $\Longrightarrow \boxed{1.}$

Problem 5. (10 points)

Problem 5.22

(a)

$$\begin{aligned}\psi(x_A, x_B, x_C) \;=\; & \frac{1}{\sqrt{6}}\left(\sqrt{\frac{2}{a}}\right)^3 \left[\sin\left(\frac{5\pi x_A}{a}\right)\sin\left(\frac{7\pi x_B}{a}\right)\sin\left(\frac{17\pi x_C}{a}\right) - \sin\left(\frac{5\pi x_A}{a}\right)\sin\left(\frac{17\pi x_B}{a}\right)\sin\left(\frac{7\pi x_C}{a}\right)\right.\\ & + \sin\left(\frac{7\pi x_A}{a}\right)\sin\left(\frac{17\pi x_B}{a}\right)\sin\left(\frac{5\pi x_C}{a}\right) - \sin\left(\frac{7\pi x_A}{a}\right)\sin\left(\frac{5\pi x_B}{a}\right)\sin\left(\frac{17\pi x_C}{a}\right)\\ & \left. + \sin\left(\frac{17\pi x_A}{a}\right)\sin\left(\frac{5\pi x_B}{a}\right)\sin\left(\frac{7\pi x_C}{a}\right) - \sin\left(\frac{17\pi x_A}{a}\right)\sin\left(\frac{7\pi x_B}{a}\right)\sin\left(\frac{5\pi x_C}{a}\right)\right].\end{aligned}$$

(b) (i)

$$\psi = \left(\sqrt{\frac{2}{a}}\right)^3 \left[\sin\left(\frac{11\pi x_A}{a}\right)\sin\left(\frac{11\pi x_B}{a}\right)\sin\left(\frac{11\pi x_C}{a}\right)\right].$$

(ii)

$$\begin{aligned}\psi \;=\; & \frac{1}{\sqrt{3}}\left(\sqrt{\frac{2}{a}}\right)^3 \left[\sin\left(\frac{\pi x_A}{a}\right)\sin\left(\frac{\pi x_B}{a}\right)\sin\left(\frac{19\pi x_C}{a}\right)\right.\\ & \left. + \sin\left(\frac{\pi x_A}{a}\right)\sin\left(\frac{19\pi x_B}{a}\right)\sin\left(\frac{\pi x_C}{a}\right) + \sin\left(\frac{19\pi x_A}{a}\right)\sin\left(\frac{\pi x_B}{a}\right)\sin\left(\frac{\pi x_C}{a}\right)\right].\end{aligned}$$

(iii)

$$\begin{aligned}\psi \;=\; & \frac{1}{\sqrt{6}}\left(\sqrt{\frac{2}{a}}\right)^3 \left[\sin\left(\frac{5\pi x_A}{a}\right)\sin\left(\frac{7\pi x_B}{a}\right)\sin\left(\frac{17\pi x_C}{a}\right) + \sin\left(\frac{5\pi x_A}{a}\right)\sin\left(\frac{17\pi x_B}{a}\right)\sin\left(\frac{7\pi x_C}{a}\right)\right.\\ & + \sin\left(\frac{7\pi x_A}{a}\right)\sin\left(\frac{17\pi x_B}{a}\right)\sin\left(\frac{5\pi x_C}{a}\right) + \sin\left(\frac{7\pi x_A}{a}\right)\sin\left(\frac{5\pi x_B}{a}\right)\sin\left(\frac{17\pi x_C}{a}\right)\\ & \left. + \sin\left(\frac{17\pi x_A}{a}\right)\sin\left(\frac{5\pi x_B}{a}\right)\sin\left(\frac{7\pi x_C}{a}\right) + \sin\left(\frac{17\pi x_A}{a}\right)\sin\left(\frac{7\pi x_B}{a}\right)\sin\left(\frac{5\pi x_C}{a}\right)\right].\end{aligned}$$

Problem 6. (10 points)

Problem 5.23

(a) $E_{n_1 n_2 n_3} = (n_1 + n_2 + n_3 + \frac{3}{2})\hbar\omega = \frac{9}{2}\hbar\omega \Rightarrow n_1 + n_2 + n_3 = 3. \quad (n_1, n_2, n_3 = 0, 1, 2, 3 \ldots).$

State n_1	n_2	n_3	Configuration $(N_0, N_1, N_2 \ldots)$	# of States
0	0	3		
0	3	0	(2,0,0,1,0,0 ...)	3
3	0	0		
0	1	2		
0	2	1		
1	0	2	(1,1,1,0,0,0 ...)	6
1	2	0		
2	0	1		
2	1	0		
1	1	1	(0,3,0,0,0 ...)	1

Possible single-particle energies:

$E_0 = \hbar\omega/2: \; P_0 = 12/30 = 4/10.$
$E_1 = 3\hbar\omega/2: \; P_1 = 9/30 = 3/10.$
$E_2 = 5\hbar\omega/2: \; P_2 = 6/30 = 2/10.$
$E_3 = 7\hbar\omega/2: \; P_3 = 3/30 = 1/10.$

Most probable configuration: (1,1,1,0,0,0 ...).

Most probable single-particle energy: $E_0 = \frac{1}{2}\hbar\omega.$

(b) For identical fermions the *only* configuration is (1,1,1,0,0,0 ...) (one state), so this is also the most probable configuration. The possible one-particle energies are

$$E_0 \; (P_0 = 1/3), \quad E_1 \; (P_1 = 1/3), \quad E_2 \; (P_2 = 1/3),$$

and they are all equally likely, so it's a 3-way tie for the most probable energy.

(c) For identical bosons all three configurations are possible, and there is one state for each. Possible one-particle energies: $E_0(P_0 = 1/3), E_1(P_1 = 4/9), E_2(P_2 = 1/9), E_3(P_3 = 1/9).$ Most probable energy: E_1.

Problem 7. (10 points)

Problem 5.32

From Problem 2.11(a),

$$\langle x\rangle_0 = \langle x\rangle_1 = 0; \quad \langle x^2\rangle_0 = \frac{\hbar}{2m\omega}; \quad \langle x^2\rangle_1 = \frac{3\hbar}{2m\omega}.$$

From Eq. 3.98,

$$\langle x\rangle_{01} = \int_{-\infty}^{\infty} x\psi_0(x)\psi_1(x)\,dx = \langle 0|x|1\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{1}\,\delta_{0\,0} + \sqrt{0}\,\delta_{1\,-1}\right) = \sqrt{\frac{\hbar}{2m\omega}}.$$

(a) Equation 5.19 $\Rightarrow \langle (x_1 - x_2)^2\rangle_d = \frac{\hbar}{2m\omega} + \frac{3\hbar}{2m\omega} - 0 = \boxed{\frac{2\hbar}{m\omega}}.$

(b) Equation 5.21 $\Rightarrow \langle (x_1 - x_2)^2\rangle_+ = \frac{2\hbar}{m\omega} - 2\frac{\hbar}{2m\omega} = \boxed{\frac{\hbar}{m\omega}}.$

(c) Equation 5.21 $\Rightarrow \langle (x_1 - x_2)^2\rangle_- = \frac{2\hbar}{m\omega} + 2\frac{\hbar}{2m\omega} = \boxed{\frac{3\hbar}{m\omega}}.$