Solutions

**Problem 1.** (20 points)

**There are questions on the back of this page.**

**(a)** [4 points] The radial component of momentum in spherical coordinates is given by

$$\hat{p}_r\psi = \frac{\hbar}{\mathrm{i}}\frac{1}{r}\frac{\partial}{\partial r}(r\psi).$$

Use this to calculate the commutator $[\hat{r}, \hat{p}_r]$.

$$[\hat{r},\hat{p}_r]\psi = \cancel{r}\frac{\hbar}{i}\frac{1}{\cancel{r}}\frac{\partial}{\partial r}(r\psi) - \frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial r}(r^2\psi)$$

$$= \frac{\hbar}{i}\left(\psi + \cancel{r\frac{\partial\psi}{\partial r}} - 2\psi - \cancel{r\frac{\partial\psi}{\partial r}}\right) = i\hbar\psi$$

$$\Rightarrow \boxed{[\hat{r},\hat{p}_r] = i\hbar}$$

**(b)** [4 points] An electron in a hydrogen atom is in the $\ell = 6$ state. What is the minimum energy it could have?

$$n \geq \ell + 1 \quad \Rightarrow \quad n_{min} = 7$$

$$\Rightarrow \boxed{E_{min} = E_7 = \frac{E_1}{49} = -\frac{13.6}{49}\,eV = -0.278\,eV}$$

**Problem 1 continued:**

An electron starts in the spin-up state, $\chi(0) = \chi_+$, at $t = 0$.

**(c)** [3 points] If $\hat{S}_x$ were measured what is the probability of getting the value $\hbar/2$?

$$\chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{so} \quad |\langle \chi_+^{(x)} | \chi_+ \rangle|^2 = \boxed{\frac{1}{2}}$$

**(d)** [4 points] If instead of making a measurement of the original spin we instead turn on a constant magnetic field in the $z$-direction, $\boldsymbol{B} = B_0\hat{k}$, what is $\chi(t)$?

In a constant B-field $E_+ = -\gamma B_0 \hbar/2$

$$\text{So} \quad \boxed{\chi(t) = e^{-iE_+t/\hbar}\chi_+ = e^{i\gamma B_0 t/2}\begin{pmatrix}1\\0\end{pmatrix}}$$

**(e)** [5 points] If $\hat{S}_x$ were measured now measured at time $t$ what is the probability of getting the value $\hbar/2$? Explain why this answer makes physical sense.

$$\text{Now} \quad \langle \chi_+^{(x)} | \chi(t) \rangle = \frac{1}{\sqrt{2}} e^{i\gamma B_0 t/2} (1, 1)\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}} e^{i\gamma B_0 t/2}$$

$$\text{So} \quad |\langle \chi_+^{(x)} | \chi(t) \rangle|^2 = \boxed{\frac{1}{2}}$$

We get the same result! This makes sense. Turning on a B-field just causes the electron to precess around the direction of the B-field. It doesn't change the alignment with respect to the x-axis.

Solutions

**Problem 2.** (30 points)
The state of an electron in a hydrogen atom is given by

$$\psi(\theta,\phi) = A\left[3\sin\theta\cos\theta\,\mathrm{e}^{\mathrm{i}\phi}\chi_+ - 2(1-\cos^2\theta)\mathrm{e}^{2\mathrm{i}\phi}\chi_-\right].$$

[***Note*: There are questions on the back of this page.**]

**(a)** [6 points] Write the state in terms of spherical harmonics and normalize it. [*Hint*: The coefficients in front of the two terms will be simple fractions.]

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\varphi}, \quad Y_2^2 = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{2i\varphi} = \frac{1}{2}\sqrt{\frac{15}{8\pi}}(1-\cos^2\theta)e^{2i\varphi}$$

Thus $\psi = A\left[-3\sqrt{\frac{8\pi}{15}}\,Y_2^1\chi_+ - 4\sqrt{\frac{8\pi}{15}}\,Y_2^2\chi_-\right]$

$= \tilde{A}\left[3Y_2^1\chi_+ + 4Y_2^2\chi_-\right]$ where I have absorbed constants into $\tilde{A}$ to simplify the normalization.

$\Rightarrow \tilde{A} = \frac{1}{5}$

So $\boxed{\psi = \left(\frac{3}{5}Y_2^1\chi_+ + \frac{4}{5}Y_2^2\chi_-\right)}$

**(b)** [4 points] If $\hat{L}_z$ were measured what values would be found and with what probabilities?

| $m_\ell$ | $L_z$ | Prob |
|---|---|---|
| 1 | $\hbar$ | 9/25 |
| 2 | $2\hbar$ | 16/25 |

**(c)** [4 points] If $\hat{S}_z$ were measured what values would be found and with what probabilities?

| $m_s$ | $S_z$ | Prob |
|---|---|---|
| 1/2 | $\hbar/2$ | 9/25 |
| -1/2 | $-\hbar/2$ | 16/25 |

**Problem 2 continued:**
The total angular momentum of this state is given by $\hat{\boldsymbol{J}} = \hat{\boldsymbol{L}} + \hat{\boldsymbol{S}}$.

**(d)** [4 points] What are the allowed values for the quantum numbers $j$ and $m_j$ for this state?

$\boxed{m_j = 3/2}$ Both terms have the same value

$\boxed{j = \frac{5}{2}, \frac{3}{2}}$

**(e)** [8 points] Expand this state in terms of the $\{|j, m_j\rangle\}$ basis. [*Hint*: Expand each term and combine the results. Your answer should already be normalized.]

Here we need to expand

$$|21\rangle |\tfrac{1}{2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{4}{5}}\, |\tfrac{5}{2} \tfrac{3}{2}\rangle - \sqrt{\tfrac{1}{5}}\, |\tfrac{3}{2}, \tfrac{3}{2}\rangle$$

$$|22\rangle |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{5}}\, |\tfrac{5}{2} \tfrac{3}{2}\rangle + \sqrt{\tfrac{4}{5}}\, |\tfrac{3}{2} \tfrac{3}{2}\rangle$$

$$\text{so } \psi = \frac{1}{5}\left[\left(3\sqrt{\tfrac{4}{5}} + 4\sqrt{\tfrac{1}{5}}\right)|\tfrac{5}{2} \tfrac{3}{2}\rangle + \left(-3\sqrt{\tfrac{1}{5}} + 4\sqrt{\tfrac{4}{5}}\, |\tfrac{3}{2} \tfrac{3}{2}\rangle\right)\right]$$

$$= \frac{1}{5}\left[10\sqrt{\tfrac{1}{5}}\, |\tfrac{5}{2} \tfrac{3}{2}\rangle + 5\sqrt{\tfrac{1}{5}}\, |\tfrac{3}{2} \tfrac{3}{2}\rangle\right]$$

$$\Rightarrow \boxed{\psi = \sqrt{\tfrac{1}{5}}\left(2|\tfrac{5}{2} \tfrac{3}{2}\rangle + |\tfrac{3}{2} \tfrac{3}{2}\rangle\right)}$$

**(f)** [4 points] If $\hat{J}^2$ were measured what values would be found and with what probabilities?

| $j$ | $\hat{J}^2$ | prob |
|---|---|---|
| $3/2$ | $\frac{15}{4}\hbar^2$ | ~~4/5~~ $1/5$ |
| $5/2$ | $\frac{35}{4}\hbar^2$ | $4/5$ |

Solutions

**Problem 3.** (15 points)
Two identical spin-$1/2$ particles, labeled 1 and 2, are separated by a distance, $\boldsymbol{a} = a\hat{z}$, and interact through a dipole coupling given by the Hamiltonian

$$\hat{H} = \frac{1}{a^3}\left[\boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_2 - \frac{3}{a^2}(\boldsymbol{\mu}_1 \cdot \boldsymbol{a})(\boldsymbol{\mu}_2 \cdot \boldsymbol{a})\right],$$

where $\boldsymbol{\mu}_j$ is the dipole moment of particle $j$. [***Note*: There is a question on the back of this page.**]

**(a)** [6 points] Write the Hamiltonian in terms of $\hat{\boldsymbol{S}}^{(1)}$ and $\hat{\boldsymbol{S}}^{(2)}$.

$\vec{\mu}_j = \gamma \hat{\vec{S}}^{(j)}$ since particles are identical $\gamma_1 = \gamma_2 \equiv \gamma$

Thus $\hat{H} = \frac{1}{a^3}\left[\gamma^2 \hat{\vec{S}}^{(1)} \cdot \hat{\vec{S}}^{(2)} - \frac{3}{a^2}\gamma^2 a^2 \hat{S}_z^{(1)} \hat{S}_z^{(2)}\right]$

$$\Rightarrow \boxed{\hat{H} = \frac{\gamma^2}{a^3}\left[\hat{\vec{S}}^{(1)} \cdot \hat{\vec{S}}^{(2)} - 3\hat{S}_z^{(1)} \hat{S}_z^{(2)}\right]}$$

**(b)** [4 points] Rewrite the Hamiltonian in terms of the total spin angular momentum operators, $\hat{S}^2$, $\hat{S}_z$, and in terms of $\left[\hat{S}^{(j)}\right]^2$ and $\hat{S}_z^{(j)}$ where $\hat{\boldsymbol{S}} = \hat{\boldsymbol{S}}^{(1)} + \hat{\boldsymbol{S}}^{(2)}$.

$\hat{S}^2 = \left(\hat{S}^{(1)}\right)^2 + \left(\hat{S}^{(2)}\right)^2 + 2\hat{\vec{S}}^{(1)} \cdot \hat{\vec{S}}^{(2)}$

So $\hat{\vec{S}}^{(1)} \cdot \hat{\vec{S}}^{(2)} = \frac{1}{2}\left[\hat{S}^2 - \left(\hat{S}^{(1)}\right)^2 - \left(\hat{S}^{(2)}\right)^2\right]$

$$\text{Thus } \boxed{\hat{H} = \frac{\gamma^2}{2a^3}\left[\hat{S}^2 - \left(\hat{S}^{(1)}\right)^2 - \left(\hat{S}^{(2)}\right)^2 - 6\hat{S}_z^{(1)} \hat{S}_z^{(2)}\right]}$$

**Problem 3 continued:**

(c) [5 points] For the two spin-$1/2$ particles find the energy of the total spin state $|0,0\rangle$.

In the $|00\rangle$ state

$\hat{S}^2$ has $S=0$ $\Rightarrow$ 0 contribution: $\hat{S}^2|00\rangle = 0$

for spin-$\frac{1}{2}$ particles $\hat{S}^{2(j)}|00\rangle = \frac{1}{2}\left(\frac{3}{2}\right)\hbar^2|00\rangle$

The individual spins are in the $\uparrow\downarrow$ configuration, thus

$$\hat{S}_z^{(1)}\hat{S}_z^{(2)}|00\rangle = \left(\frac{\hbar}{2}\right)\left(-\frac{\hbar}{2}\right)|00\rangle = -\frac{\hbar^2}{4}|00\rangle.$$

Thus $\hat{H}|00\rangle = \frac{\gamma^2}{2a^3}\left[0 - \frac{3}{2}\hbar^2 + \frac{6}{4}\hbar^2\right]|00\rangle = 0$

Thus $\boxed{E=0}$