Problem 1. (28 points) Solutions

Clearly explain your answers to the following questions. **There are questions on the back of this page.**

(a) [6 points] For some operator, $\hat{B}$, let $\hat{B}|\psi_n\rangle = 2^n|\psi_n\rangle$. Given the state,

$$|\psi\rangle = A\left(\frac{1}{\sqrt{2}}|\psi_0\rangle + \frac{1}{\sqrt{5}}|\psi_1\rangle + \frac{1}{\sqrt{10}}|\psi_2\rangle\right)$$

Find A and calculate $\langle\hat{B}\rangle$.

$$\langle\psi|\psi\rangle = 1 = |A|^2\left(\frac{1}{2}+\frac{1}{5}+\frac{1}{10}\right) = |A|^2\left(\frac{5+2+1}{10}\right) = |A|^2\left(\frac{8}{10}\right)$$

$$\Rightarrow \boxed{A = \sqrt{\frac{5}{4}}}$$

$$\langle\hat{B}\rangle = \frac{5}{4}\left[\frac{1}{2}(1) + \frac{1}{5}(2) + \frac{1}{10}(4)\right] = \frac{5}{4}\left[\frac{5+4+4}{10}\right] = \frac{1}{4}\left(\frac{13}{2}\right)$$

$$\Rightarrow \boxed{\langle\hat{B}\rangle = \frac{13}{8}}$$

(b) [4 points] In a quantum mechanics book I found the following statement in a problem: Consider three observables, $\hat{A}$, $\hat{B}$, and $\hat{C}$ that satisfy $[\hat{B},\hat{C}] = \hat{A}$. Explain why these three operators *cannot* all be observables.

To be an observable ~~all~~ an operator must be Hermitian.

Suppose $\hat{B}^\dagger = \hat{B}$ and $\hat{C}^\dagger = \hat{C}$. Then $\hat{A}^\dagger = [\hat{B},\hat{C}]^\dagger = (\hat{B}\hat{C}-\hat{C}\hat{B})^\dagger = \hat{C}^\dagger\hat{B}^\dagger - \hat{B}^\dagger\hat{C}^\dagger$

$= \hat{C}\hat{B} - \hat{B}\hat{C} = -[\hat{B},\hat{C}] = -\hat{A}$.

Thus $\hat{A}$ cannot be Hermitian.

(c) [6 points] What the book referenced in the previous problem meant to say was that for these operators suppose that $[\hat{B},\hat{C}] = i\hat{A}$ and $[\hat{A},\hat{C}] = i\hat{B}$. Now calculate the uncertainty relation for $\sigma_{AB}\sigma_C$.

$$\sigma_{AB}\sigma_C \geq \left|\frac{1}{2i}\langle[\hat{A}\hat{B},\hat{C}]\rangle\right|. \qquad \hat{B}\hat{C} = i\hat{A} + \hat{C}\hat{B}$$

$$[\hat{A}\hat{B},\hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B} = \hat{A}(i\hat{A} + \hat{C}\hat{B}) - \hat{C}\hat{A}\hat{B}$$

$$= i\hat{A}^2 + \hat{A}\hat{C}\hat{B} - \hat{C}\hat{A}\hat{B} = i\hat{A}^2 + [\hat{A},\hat{C}]\hat{B} = i\hat{A}^2 + i\hat{B}^2.$$

Thus $\boxed{\sigma_{AB}\sigma_C \geq \frac{1}{2}\left[\langle\hat{A}^2\rangle + \langle\hat{B}^2\rangle\right]}$

Problem 1 continued:

(d) [8 points] Find the eigenfunctions, $\psi(x)$, of the ladder operator, $\hat{a}_+$, defined for the simple harmonic oscillator. What are the allowed eigenvalues? [*Hint*: You should find a first order differential equation that is easy to integrate.]

$$\hat{a}_+\psi = \alpha\psi \Rightarrow \frac{\cancel{1}}{\cancel{\sqrt{2}}} \frac{1}{\sqrt{2m\hbar\omega}}\left(-\hbar\frac{d}{dx} + m\omega x\right)\psi = \alpha\psi.$$

$$\Rightarrow + \cancel{\hbar}\frac{d\psi}{dx} = \frac{1}{\hbar}\left(m\omega x - \alpha\sqrt{2m\hbar\omega}\right)\psi$$

So $$\boxed{\psi(x) = A\exp\left\{\frac{1}{\hbar}\left(\frac{m\omega}{2}x^2 - \alpha\sqrt{2m\hbar\omega}\,x\right)\right\}}$$

The eigen value $\underline{\alpha \in \mathbb{C}}$. $\hat{a}_+$ is NOT Hermitian so it doesn't have to have real eigenvalues.

(e) [4 points] Are the eigenfunctions you found in the previous part also eigenfunctions of $\hat{a}_-$? [*Note*: You can either explain whether they can or cannot be or show the result by explicit calculation.]

$\psi(x)$ from (d) are NOT eigenfunctions of $\hat{a}_-$.
Since $[\hat{a}_-, \hat{a}_+] = 1 \neq 0$ They are not compatible observables so we cannot find ~~eigen~~functions that are eigenfunctions of both operators.

Solutions

Problem 2. (32 points)

The states $\{|1\rangle, |2\rangle, |3\rangle\}$ form a basis in a three dimensional Hilbert space. Consider the Hamiltonian

$$\hat{H} = |1\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 1| - |2\rangle\langle 3| - |3\rangle\langle 2| + |3\rangle\langle 3|.$$

[*Note*: **There are questions on the back of this page.**]

(a) [4 points] Write the Hamiltonian as a matrix and verify that it is Hermitian.

$H_{mn} = \langle m|\hat{H}|\hat{n}\rangle$.

Thus $\boxed{H = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}}$ Clearly $\underline{H^\dagger = H}$.

(b) [10 points] Find the eigenvalues, $\{E_i\}$, and normalized eigenvectors, $\{|e_i\rangle\}$ for $\hat{H}$. Make sure it is clear which eigenvalue corresponds to which eigenvector.

$$|H - EI| = 0 = \begin{vmatrix} 1-E & 1 & 0 \\ 1 & -E & -1 \\ 0 & -1 & 1-E \end{vmatrix} = (1-E)\begin{vmatrix} -E & -1 \\ -1 & 1-E \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ 0 & 1-E \end{vmatrix}$$

$$= [-E(1-E)-1](1-E) - (1-E) = 0$$

$$\Rightarrow (1-E)[-E+E^2-2] = (1-E)(E^2-E-2) = -(E-1)(E-2)(E+1) = 0$$

Thus $\boxed{E_2 = 2,\ E_\pm = \pm 1}$ are the eigenvalues.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = E\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \Rightarrow \begin{matrix} \alpha+\beta = E\alpha & \Rightarrow \beta = (E-1)\alpha \\ \alpha - \gamma = E\beta & \\ -\beta+\gamma = E\gamma & \Rightarrow \beta = (1-E)\gamma. \end{matrix}$$

$\boxed{E_2 = 2}$: Let $\alpha = 1 \Rightarrow \beta = 1, \gamma = -1$

$$|e_2\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$\boxed{E_+ = +1}$: Let $\alpha = 1 \Rightarrow \beta = 0, \gamma = 1$

$$|e_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$\boxed{E_- = -1}$: Let $\alpha = 1$, $\beta = -2$, $\gamma = -1$

$$|e_-\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

Problem 2 continued:

(c) [6 points] Suppose the system starts in the state

$$|\mathscr{S}(0)\rangle = \begin{pmatrix} 0 \\ \mathrm{i} \\ 0 \end{pmatrix}.$$

Write $|\mathscr{S}(0)\rangle$ in the original basis $\{|1\rangle, |2\rangle, |3\rangle\}$ and in the basis of the eigenstates of the Hamiltonian, $\{|e_i\rangle\}$

$$\boxed{|S(0)\rangle = i\,|2\rangle}$$

$$\langle e_2|S(0)\rangle = \frac{i}{\sqrt{3}}$$

$$\langle e_+|S(0)\rangle = 0$$

$$\langle e_-|S(0)\rangle = \frac{-2i}{\sqrt{6}} = -\sqrt{\frac{2}{3}}\,i$$

Thus

$$\boxed{|S(0)\rangle = \frac{i}{\sqrt{3}}|e_2\rangle - i\sqrt{\frac{2}{3}}\,|e_-\rangle}$$

(d) [8 points] Write $|\mathscr{S}(t)\rangle$ in both bases.

$$\boxed{|S(t)\rangle = \frac{i}{\sqrt{3}}|e_2\rangle e^{-i2t/\hbar} - i\sqrt{\frac{2}{3}}\,|e_-\rangle e^{it/\hbar}}$$

← eigenstates of $\hat{H}$, so t evolution easy.

So

$$\boxed{|S(t)\rangle = \frac{i}{3}\begin{pmatrix} e^{-i2t/\hbar} - e^{it/\hbar} \\ e^{-i2t/\hbar} + 2e^{it/\hbar} \\ -e^{-i2t/\hbar} + e^{it/\hbar} \end{pmatrix}}$$

(e) [4 points] Calculate the probability of measuring the system in the state $|\mathscr{S}(0)\rangle$ as a function of time. Verify that you get the correct answer at $t = 0$.

$$\boxed{P(t) = |\langle S(0)|S(t)\rangle|^2 = \left|\frac{1}{3}\left(e^{-i2t/\hbar} + 2e^{it/\hbar}\right)\right|^2.}$$

For $t=0$: $P(0) = \left|\frac{1}{3}(1+2)\right|^2 = 1$ as it must.