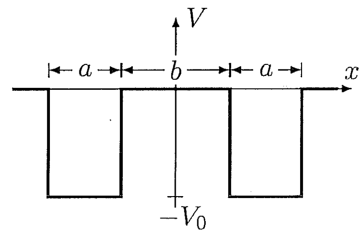
Solutions



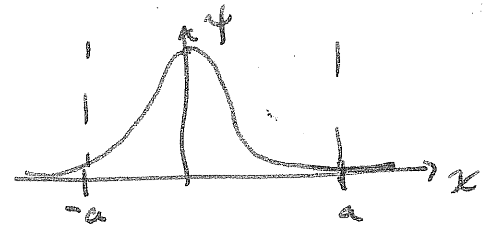
Problem 1. (25 points)
Consider the double well potential shown at the right. Assume that the well is deep enough to allow for multiple bound states. **This probably is qualitative, you should NOT solve the Schrödinger equation!** [*Note*: **There are questions on the back of this page.**]

(a) [2 points] Is the ground state wave function, ψ_1, even or odd? How many nodes does it have?

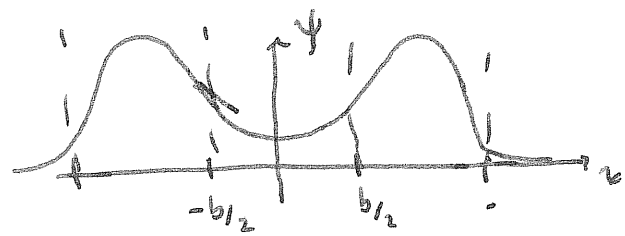
Even with 0 nodes.

(b) [8 points] Sketch the ground state wave function, ψ_1, for the cases when (i) $b = 0$, (ii) $b \approx a$, and (iii) $b \gg a$.

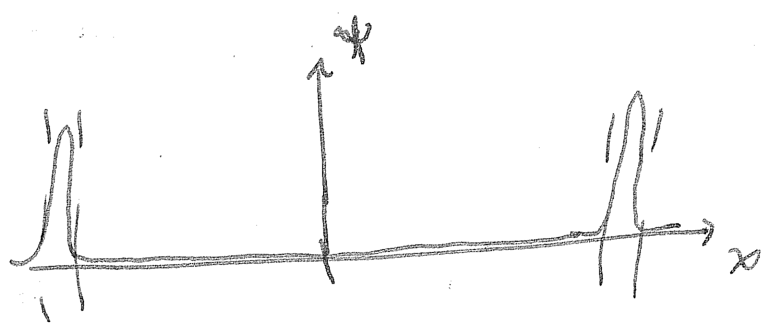
i) $b=0$: just a square well



ii) $b \approx a$: intermediate state



iii) $b \gg a$: like two separate wells.

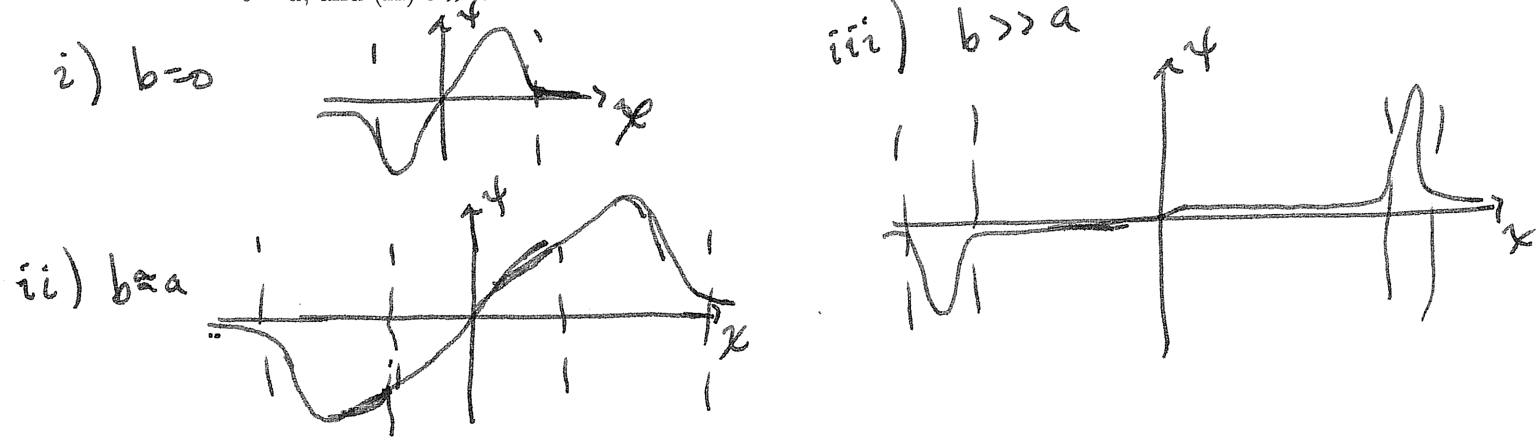


Problem 1 continued:

(c) [2 points] Is the first excited state wave function, ψ_2, even or odd? How many nodes does it have?

Odd with 1 node. Due to symmetry this node is at $x=0$

(d) [8 points] Sketch the first excited state wave function, ψ_2, for the cases when (i) $b=0$, (ii) $b \approx a$, and (iii) $b \gg a$.



(e) [5 points] Qualitatively, how does the energies of these two states, E_1 and E_2, vary as b goes from 0 to ∞? Sketch $E_1(b)$ and $E_2(b)$ on the same graph. [*Hint*: Use your plots of the wave functions as a guide. For $b=0$ we know the result, for $b \gg a$ what do the wave functions look like and how does ψ_1 compare to ψ_2?]

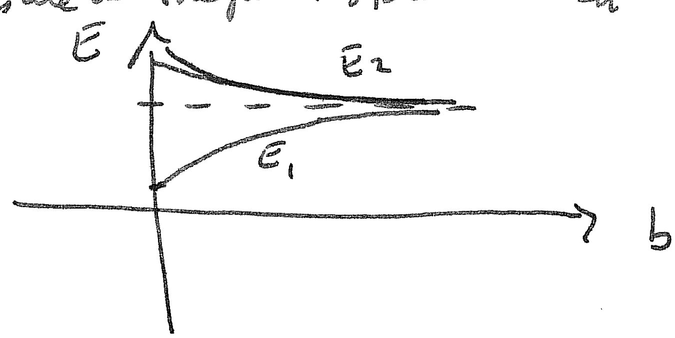
For a square well $E_n \propto \frac{n^2}{a^2}$

When $b=0$ the width of the well is $2a$ so $E_n \sim \frac{n^2}{4a^2}$.

When $b \gg a$ then we have two separate wells. From our plots it looks like both particles are in the ground state $\Rightarrow E_n \sim \frac{1}{a^2}$.

Thus $E_1 \sim \frac{1}{4a^2} \to \frac{1}{a^2}$

$E_2 \sim \frac{1}{a^2} \to \frac{1}{a^2}$.



Solutions

Problem 2. (20 points)
A particle is placed in an infinite potential well where $V = 0$ for $x \in [0, a]$ as discussed in class. Let the initial wave function be

$$\Psi(x,0) = \begin{cases} 0, & x < 0 \\ Ax, & 0 < x < a \\ 0, & x > a \end{cases} .$$

[*Note*: **There are questions on the back of this page.**]

(a) [3 points] Find the normalization constant A.

$$\int |\Psi|^2 dx = 1 \Rightarrow \int_0^a |A|^2 x^2 dx = 1 = \frac{a^3}{3}|A|^2$$

$$\Rightarrow \boxed{A = \sqrt{\frac{3}{a^3}}}$$

(b) [8 points] Expand the initial wave function in terms of the solutions to the time independent Schrödinger equation, ψ_n. That is write

$$\Psi(x,0) = \sum_n c_n \psi_n(x)$$

and find c_n.

$$c_n = \int \Psi(x,0)\psi_n(x)dx = \sqrt{\frac{3}{a^3}} \int_0^a x \sqrt{\frac{2}{a}} \sin\left(n\frac{\pi x}{a}\right) dx$$

$$= \frac{\sqrt{6}}{a^2} \int_0^a x \sin\left(n\frac{\pi x}{a}\right) dx$$

$$= \frac{\sqrt{6}}{a^2} \left[\frac{a^2}{n^2\pi^2} \sin\left(\frac{n\pi x}{a}\right) - \frac{a x}{n\pi} \cos\left(n\frac{\pi x}{a}\right) \right|_0^a$$

$$= \frac{\sqrt{6}}{a^2} \left(0 - \frac{a^2}{n\pi} \cos n\pi \right) = \frac{\sqrt{6}}{n\pi} (-1)^{n+1}$$

$$\Rightarrow \cancel{\boxed{c_n = \frac{6}{\pi n}}} \quad \boxed{c_n = \frac{\sqrt{6}}{\pi} \frac{1}{n} (-1)^{n+1}}$$

Problem 2 continued:

(c) [4 points] Using the fact that

$$\zeta(2) \equiv \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

verify that we can treat the $|c_n|^2$ as probabilities.

To be probabilities $\sum |c_n|^2 = 1$

So $\displaystyle\sum_{n=1}^{\infty} \left| \frac{\sqrt{6}}{\pi} \frac{(-1)^n}{n} \right|^2 = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{6}{\pi^2} \zeta(2) = \underline{1}$ as required.

(d) [5 points] Calculate the expectation value for the energy. You should get an unphysical result. Why do you think this happens?

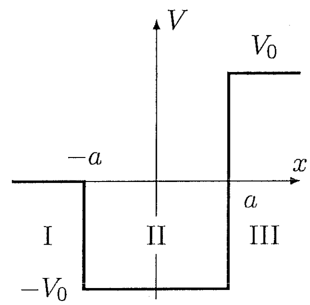
$$\langle \hat{H} \rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n = \sum_{n=1}^{\infty} \left(\frac{6}{\cancel{n^2}\cancel{\pi^2}}\right)\left(\frac{\cancel{n^2}\cancel{\pi^2}\hbar^2}{2ma^2}\right)$$

$$= \sum_{n=1}^{\infty} \left(\frac{6\hbar^2}{2ma^2}\right) = \underline{\underline{\infty}} \quad !$$

Our initial wave function $\Psi(x,0)$ does not satisfy the boundary conditions, $\Psi(a,0) = a \neq 0$. We can still expand this function in ψ_n but we see it would take an infinite amount of energy to actually make the state.

Problem 3. (30 points) Solutions

Consider the finite square well/square barrier potential shown at the right. [*Note*: **There are questions on the back of this page.**]



(a) [6 points] A beam of particles is sent in from the left and treated as a plane wave with $E > V_0$. Write down the solution to the Schrödinger equation in each region. Also write down an expression for the wave number, k, in each region.

$$\psi_I = Ae^{ik_1x} + Be^{-ik_1x} \qquad \text{where} \quad k_1 = \frac{\sqrt{2mE}}{\hbar}$$

$$\psi_{II} = Ce^{ik_2x} + De^{-ik_2x} \qquad k_2 = \frac{\sqrt{2m(E+V_0)}}{\hbar}$$

$$\psi_{III} = Fe^{ik_3x} \qquad k_3 = \frac{\sqrt{2m(E-V_0)}}{\hbar}$$

only left moving wave in III

(b) [8 points] Apply the appropriate boundary conditions and write down the conditions that the constants from part (a) must satisfy. Do **not** solve these equations.

① ψ is continuous

$$\Rightarrow \psi_I(-a) = \psi_{II}(-a) : \quad Ae^{-ik_1a} + Be^{ik_1a} = Ce^{-ik_2a} + De^{ik_2a}$$

$$\psi_{II}(a) = \psi_{III}(a) : \quad Ce^{ik_2a} + De^{-ik_2a} = Fe^{ik_3a}$$

② ψ' is continuous

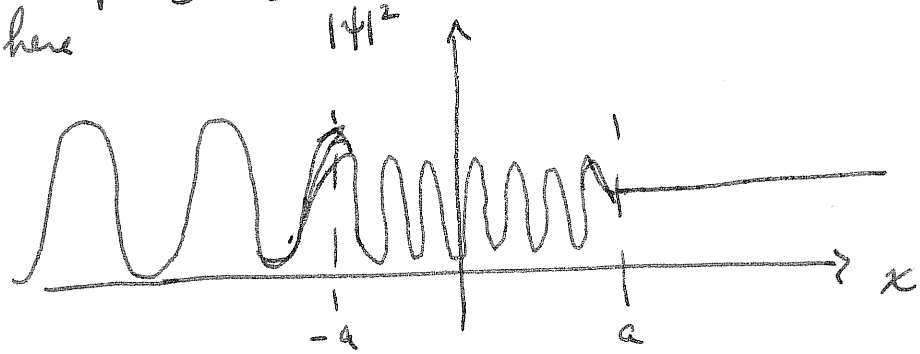
$$\Rightarrow \psi_I'(-a) = \psi_{II}'(-a) : \quad k_1\left(Ae^{-ik_1a} - Be^{ik_1a}\right) = k_2\left(Ce^{-ik_2a} - De^{ik_2a}\right)$$

$$\psi_{II}'(a) = \psi_{III}'(a) : \quad k_2\left(Ce^{ik_2a} - De^{-ik_2a}\right) = k_3Fe^{ik_3a}.$$

Problem 3 continued:

(c) [7 points] Sketch the probability density for the particles considered in part (a). Make sure the important physical features are clear in your sketch.

$k_2 > k_1 \Rightarrow$ frequency higher in region II. Also particle spends less time here



(d) [3 points] We know that treating a beam of particles as a plane wave is incorrect, why?

$e^{\pm ikx}$ is not a normalizable wave fcn so cannot describe a physical state.

(e) [6 points] We know that we should treat a free particle as a wave packet. Consider a Gaussian wave packet that start at rest at $x = 0$. Qualitatively describe its behavior. Describe what happens when the wave packet starts to evolve and what the long time probability distribution will look like.

The wave packet will spread out in time. It will spread faster in the $x<0$ direction so the probability density will be skewed to that side. Presumably there will also be some probability of being trapped in the well. The probability at $x=0$ will not be 0.