Problem 1. (30 points)

Clearly justify your answers to the following questions. [***Note*: There are questions on the back of this page.**]

(a) [3 points] What are the units of a one dimensional wavefunction, ψ?

(b) [4 points] Why are observations so important to the formulation of quantum mechanics? How does an observation affect a quantum mechanical state?

(c) [4 points] The book states, in bold, that "observables are represented by Hermitian operators". Why is this true?

(d) [4 points] What are "stationary states" and why are they important?

Problem 1 continued:
Are the following states valid wavefunctions for an electron in a hydrogen atom? Justify your answers.

(e) [3 points] $|\psi\rangle = R_{3,2}Y_1^0 \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$.

(f) [3 points] $|\psi\rangle = R_{3,2}Y_2^0 \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$.

(g) [3 points] $|\psi\rangle = R_{3,3}Y_3^1 \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$.

(h) [3 points] $|\psi\rangle = R_{3,2}Y_2^3 \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$.

(i) [3 points] $|\psi\rangle = R_{3,2}Y_2^2 \left|-\frac{1}{2}, \frac{1}{2}\right\rangle$.

Problem 2. (26 points)

(a) [4 points] Calculate the commutator $[\hat{L}_z, \sin(2\varphi)]$.

(b) [8 points] The Hamiltonian for an axially symmetric rotator is given by

$$\hat{H} = \frac{\hat{L}_x^2 + \hat{L}_y^2}{2I_1} + \frac{\hat{L}_z^2}{2I_2}$$

where the I_j are constant moments of inertia. Determine the eigenvalues and eigenstates of this Hamiltonian.

Problem 2 continued:

(c) [4 points] One formulation of classical mechanics uses the Poisson bracket defined (in one dimension) for two functions $u(x, p)$ and $v(x, p)$ by

$$\{u, v\} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial p} - \frac{\partial u}{\partial p}\frac{\partial v}{\partial x}.$$

Calculate $\{x, p\}$ and compare this to the commutator in quantum mechanics. [*Note*: One means of transitioning from classical mechanics to quantum mechanics is through this relationship between the Poisson bracket and the commutator.]

(d) [6 points] Consider the Hermitian operators $\hat{A}$ and $\hat{B}$. Suppose that $\hat{A}^2 = 2\hat{1}$ and $\hat{B}^4 = \hat{1}$ where $\hat{1}$ is the identity operator. What are the eigenvalues of these operators and what are their degeneracies?

(e) [4 points] Suppose the operators in the previous part were not Hermitian. Now determine their eigenvalues and degeneracies.

Problem 3. (24 points)

For the simple harmonic oscillator the eigenstates of the Hamiltonian are given by $\{|n\rangle\}$. [*Note*: **There are questions on the back of this page.**]

(a) [12 points] Calculate matrix representations of $\hat{x}$ and $\hat{p}$ in this basis. Also calculate the matrix representations for the products $\hat{x}\hat{p}$ and $\hat{p}\hat{x}$. These are infinite dimensional matrices so calculate enough terms so that the pattern is clear.

Problem 3 continued:

(b) [2 points] Using your matrix representations to verify the commutation relation for $\hat{x}$ and $\hat{p}$.

(c) [6 points] Calculate $\langle \hat{x}^2 \rangle$ using the matrix representation for $\hat{x}$ to first calculate the representation for $\hat{x}^2$ then use this to find the desired expectation value.

(d) [4 points] Calculate $\langle \hat{x}^2 \rangle$ using the fact that $\sum_{m=0}^{\infty} |m\rangle \langle m| = \hat{1}$. Thus first write

$$\langle \hat{x}^2 \rangle = \sum_{m=0}^{\infty} \langle n|\hat{x}|m\rangle \langle m|\hat{x}|n\rangle \,,$$

and proceed with the calculation.

Problem 4. (24 points)

The Hamiltonian for a particle of mass, m, and charge, q, in a constant electric field, $\mathscr{E}_0$, is given by

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 - q\mathscr{E}_0\hat{x}.$$

[***Note*: There are questions on the back of this page.**]

(a) [8 points] Make the transformation

$$\hat{\eta} = \hat{x} - \frac{q\mathscr{E}_0}{m\omega^2}$$

to rewrite the Hamiltonian as one you recognize. Write down the eigenstates for this Hamiltonian.

(b) [3 points] Calculate the energy levels for this system.

Problem 4 continued:

(c) [8 points] Suppose we instead have a time dependent electric field so that the Hamiltonian is now

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 - q\mathscr{E}_0\cos(\Omega t)\hat{x}.$$

Solving this problem is more challenging! Instead of solving it calculate the following derivatives of expectation values

$$\frac{\mathrm{d}\langle\hat{x}\rangle}{\mathrm{d}t}, \quad \frac{\mathrm{d}\langle\hat{p}\rangle}{\mathrm{d}t}, \quad \frac{\mathrm{d}\langle\hat{H}\rangle}{\mathrm{d}t}.$$

[*Note*: Your answers may involve expectation values of operators.]

(d) [5 points] Solve for $\langle\hat{x}\rangle(t)$. [*Hint*: Write down a second order differential equation for $\langle\hat{x}\rangle$.]

Problem 5. (30 points)

Consider the double Dirac delta potential

$$V(x) = -\alpha\delta(x+a) + \alpha\delta(x-a)$$

where α and a are positive constants. **There are questions on the back of this page.**]

(a) [8 points] For the case $E < 0$ write down the wavefunction in terms of unknown normalization constants. Clearly define any constants and regions you use for breaking up your wavefunction.

(b) [10 points] Write down the conditions the normalization constants from the previous part must satisfy. Clearly identify the source of these conditions.

Problem 5 continued:

(c) [6 points] For a plane wave sent in from $x < 0$ with $E > 0$ write down the wavefunction in terms of unknown normalization constants. Clearly define any constants and regions you use for breaking up your wavefunction.

(d) [6 points] Write down the conditions the normalization constants from the previous part must satisfy. Clearly identify the source of these conditions.

Problem 6. (36 points)
Two identical, non-interacting particles each of mass $m > 0$ are placed in an infinite square well potential of width a in the state

$$\psi = \psi_{\text{space}}\psi_{\text{spin}} = \frac{\sqrt{2}}{a}\left[\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{5\pi x_2}{a}\right) + \sin\left(\frac{5\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right)\right]\psi_{\text{spin}},$$

where ψ_{spin} is the state of the spins of the system. For the following problems determine if this is a valid state for the system and write down all the allowed spin states, ψ_{spin}, in terms of the total spin and its z-projection quantum numbers, $|s, m_s\rangle$. [*Note*: **There are questions on the back of this page.**]

(a) [2 points] Is ψ_{space} symmetric or antisymmetric under the interchange of the two particles?

(b) [4 points] Spin-0 particles.

(c) [4 points] Spin-1/2 particles.

(d) [8 points] Spin-1 particles.

Problem 6 continued:

Suppose we instead put two particles in the state

$$\psi = \psi_{\text{space}}\psi_{\text{spin}} = \frac{\sqrt{2}}{a}\left[\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{5\pi x_2}{a}\right) - \sin\left(\frac{5\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right)\right]\psi_{\text{spin}}.$$

Reconsider the previous cases for this state.

(e) [2 points] Is ψ_{space} symmetric or antisymmetric under the interchange of the two particles?

(f) [4 points] Spin-0 particles.

(g) [6 points] Spin-1/2 particles.

(h) [6 points] Spin-1 particles.