Solutions

Problem 1. (20 points)

There are questions on the back of this page.

(a) [5 points] Calculate the uncertainty relation between $\hat{S}^2$ and $\hat{S}_x$ for an electron in the χ_- state.

$[\hat{S}^2, \hat{S}_x] = 0$ So these are compatible observables.

Thus $\boxed{\sigma_{S^2} \sigma_{S_x} \geq 0}$

(b) [5 points] Calculate the uncertainty relation between $\hat{S}_x$ and $\hat{S}_y$ for an electron in the χ_- state.

$[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z$.

$\sigma_{S_x} \sigma_{S_y} \geq \left| \frac{1}{2i} \langle [\hat{S}_x, \hat{S}_y] \rangle \right|$.

$\langle \hat{S}_z \rangle = \langle \chi_- | \hat{S}_z | \chi_- \rangle = -\frac{\hbar}{2}$.

Thus $\sigma_{S_x} \sigma_{S_y} \geq \left| \frac{1}{2\cancel{i}} (\cancel{i}\hbar) \left(-\frac{\hbar}{2}\right) \right|$

$\Rightarrow \boxed{\sigma_{S_x} \sigma_{S_y} \geq \frac{\hbar^2}{4}}$

Problem 1 continued:

(c) [5 points] Suppose we were to model the Hydrogen atom as an infinite spherical well with radius $a = a_0$ where a_0 is the Bohr radius. A "spin-down" electron is placed in the well. Write down the *complete ground state wave function* $\psi(\boldsymbol{r})$ for the electron in terms of r, θ, ϕ, a_0, spin states, and constants, as appropriate.

Using the result of the wavefunctions for the infinite spherical well and the fact that the ground state has $n=1,\ l=0,\ m_l=0$ we have

$$\boxed{\psi_{gs}(\vec{r}) = \frac{1}{\sqrt{2\pi a_0}} \frac{\sin(\pi r/a_0)}{r} \chi_-}$$

(d) [5 points] To test the validity of the model in previous problem calculate the ratio of the magnitude of the ground state energy of the infinite spherical well to that of the Hydrogen atom, $|E_{SW,gs}/E_{H,gs}|$. [*Hint*: If you are smart about how you write the energies no numerical values are required.]

For the spherical well: $E_{SW,gs} = \dfrac{\pi^2\hbar^2}{2m a_0^2}$

For the Hydrogen atom: $|E_{H,gs}| = \dfrac{\hbar^2}{2m a_0^2}$

Thus $$\boxed{\left|\frac{E_{SW,gs}}{E_{H,gs}}\right| = \pi^2}$$

Solutions

Problem 2. (30 points)
A particle with mass M is constrained to move on the surface of a sphere with radius R. It can move freely on the sphere but not off the sphere. The Hamiltonian for this system is

$$\hat{H} = \frac{\hat{L}^2}{2MR^2}.$$

[*Note*: **There are questions on the back of this page.**]

(a) [10 points] Write down the eigenvalues and eigenstates for this Hamiltonian. What is the degeneracy of the eigenvalues? [*Hint*: You can just "write down" the answers with appropriate explanations.]

The Hamiltonian is just proportional to $\hat{L}^2$, thus the eigenstates of $\hat{H}$ are the same as those of $\hat{L}^2$.

Thus $\boxed{\text{Eigenstates: } Y_\ell^m(\theta, \varphi)}$

$\hat{L}^2 Y_\ell^m = \ell(\ell+1)\hbar^2 Y_\ell^m$ so $\boxed{\text{eigenvalues: } \frac{\ell(\ell+1)\hbar^2}{2MR^2}}$

The eigenvalues are independent of m_ℓ so $\boxed{\text{degeneracy: } 2\ell+1}$

(b) [5 points] Suppose the particle starts in the state

$$\psi(\boldsymbol{r}) = \frac{1}{\sqrt{8\pi}} + \sqrt{\frac{15}{32\pi}} \cos 2\phi \sin^2\theta.$$

Write this state in terms of the eigenstates of the Hamiltonian.

$Y_0^0 = \frac{1}{\sqrt{4\pi}}$, $Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta \, e^{\pm 2i\varphi}$, $\cos 2\varphi = \frac{1}{2}\left(e^{2i\varphi} + e^{-2i\varphi}\right)$.

Thus $\boxed{\psi(r) = \frac{1}{\sqrt{2}} Y_0^0 + \frac{1}{2}\left(Y_2^{-2} + Y_2^{2}\right)}$

Problem 2 continued:

(c) [5 points] Suppose we measure the energy for the state in part (b), what values can we get and with what probabilities?

We have values $\ell = 0$ or $\ell = 2$ so we get

$$\boxed{\begin{array}{llll} \ell = 0, & E = 0, & \text{prob} = 1/2 \\ \ell = 2, & E = \frac{3\hbar^2}{MR^2}, & \text{prob} = 1/2 \end{array}}$$

(d) [5 points] Suppose we measure the square of the angular momentum for the state in part (b), what values can we get and with what probabilities?

As above we have $\ell = 0$ or $\ell = 2$.

So

$$\boxed{\begin{array}{llll} \ell = 0, & 0, & \text{prob} = \frac{1}{2} \\ \ell = 2, & 6\hbar^2, & \text{prob} = \frac{1}{2} \end{array}}$$

(e) [5 points] Suppose we measure the z-component of the angular momentum for the state in part (b), what values can we get and with what probabilities?

Here we can measure $m_\ell = 0, \pm 2$.

So

$$\boxed{\begin{array}{llll} m_\ell = 0, & L_z \to 0, & \text{prob} = \frac{1}{2} \\ m_\ell = 2, & L_z \Rightarrow 2\hbar, & \text{prob} = 1/4 \\ m_\ell = -2, & L_z \to -2\hbar, & \text{prob} = \frac{1}{4} \end{array}}$$

Problem 3. (25 points) Solutions

The wave function for an electron in a hydrogen atom is

$$\psi(\boldsymbol{r}) = R_{32}(r)\left[\frac{2}{\sqrt{5}}Y_2^1(\theta,\phi)\left|\frac{1}{2},\frac{1}{2}\right\rangle - \frac{1}{\sqrt{5}}Y_2^2(\theta,\phi)\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right].$$

The total momentum operator is $\hat{\boldsymbol{J}} = \hat{\boldsymbol{L}} + \hat{\boldsymbol{S}}$.

(a) [4 points] Is this wave function an eigenstate of $\hat{J}_z$? If yes find its eigenvalue.

$\boxed{\text{Yes}}$ For this state $m_j = 3/2$ for both terms

Thus $\boxed{\text{eigenvalue is } \frac{3}{2}\hbar}$

(b) [4 points] If we measure the z-component of the electron's spin what values can we get and with what probabilities?

Measurement gives

$$\boxed{\begin{array}{llll} m_s = & \frac{1}{2}, & S_z = \frac{\hbar}{2}, & \text{prob} = \frac{4}{5} \\ & -\frac{1}{2} & -\frac{\hbar}{2} & \frac{1}{5} \end{array}}$$

(c) [4 points] If we measure $\hat{J}^2$ what values can we get?

$j = |l-s|, \ldots, l+s$

So $\boxed{j = \frac{3}{2} \text{ or } 5/2}$ or equivalently we would measure $\underline{\frac{15}{4}\hbar^2 \text{ or } \frac{35}{4}\hbar^2}$

Problem 3 continued:

(d) [9 points] Rewrite the wave function in terms of the basis states $\{|j, m_j\rangle\}$ of the total angular momentum operator, $\hat{\boldsymbol{J}}$.

We have $\ell = 2$, $s = \frac{1}{2}$ and 2 states to expand

$$|2\,1\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle = \sqrt{\tfrac{4}{5}}\,|\tfrac{5}{2}\,\tfrac{3}{2}\rangle - \tfrac{1}{\sqrt{5}}|\tfrac{3}{2}\,\tfrac{3}{2}\rangle$$

$$|2\,2\rangle|\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{5}}|\tfrac{5}{2}\,\tfrac{3}{2}\rangle + \sqrt{\tfrac{4}{5}}\,|\tfrac{3}{2}\,\tfrac{3}{2}\rangle$$

Thus

$$\psi(\vec{r}) = R_{32}\Big[\tfrac{2}{\sqrt{5}}\Big(\tfrac{2}{\sqrt{5}}|\tfrac{5}{2}\,\tfrac{3}{2}\rangle - \tfrac{1}{\sqrt{5}}|\tfrac{3}{2}\,\tfrac{3}{2}\rangle\Big) - \tfrac{1}{\sqrt{5}}\Big(\tfrac{1}{\sqrt{5}}|\tfrac{5}{2}\,\tfrac{3}{2}\rangle + \sqrt{\tfrac{4}{5}}\,|\tfrac{3}{2}\,\tfrac{3}{2}\rangle\Big)\Big]$$

So

$$\boxed{\psi(\vec{r}) = R_{32}(r)\Big[\tfrac{3}{5}|\tfrac{5}{2}\,\tfrac{3}{2}\rangle - \tfrac{4}{5}|\tfrac{3}{2}\,\tfrac{3}{2}\rangle\Big]}$$

(e) [4 points] What are the probabilities of measuring the values of $\hat{J}^2$ you found in part (c)?

From the previous part we see that

$$\boxed{\begin{array}{ll} j = \tfrac{5}{2}, & \text{prob} = \tfrac{9}{25} \\ j = \tfrac{1}{2}, & \text{prob} = \tfrac{16}{25} \end{array}}$$