

Problem 1. (25 points)

Solutions

Clearly explain your answers to the following questions. There are questions on the back of this page.

- (a) [4 points] In quantum mechanics states in position space are represented by normalized functions, $\psi(x)$. Does the set of all normalized functions form a vector space?

$$\langle \psi(x) | \psi(x) \rangle = 1$$

Not a vector space

Consider ψ_1, ψ_2 ; then $\psi_1 + \psi_2$ is not normalized.

Similarly $a\psi_1$ is also not normalized.

Thus $\psi_1 + \psi_2$ and $a\psi_1$ are not in the set of normalized functions.

- (b) [4 points] Let \mathcal{H} be the Hilbert space of functions on the real line, $-\infty < x < \infty$. Is the function $f(x) = e^{-kx}$ where k is any real number an element of this Hilbert space, \mathcal{H} ?

To be in Hilbert space the function must be normalizable.

e^{-kx} is not normalizable.

Thus $e^{-kx} \notin \mathcal{H}$

- (c) [5 points] A particle is moving in a constant potential, $V(x) = V_0$. Can the momentum and total energy of this particle simultaneously be measured to arbitrary precision?

To be able to measure them simultaneously we need $\sigma_H \sigma_p = 0$

Which means they commute, $[\hat{H}, \hat{p}] = 0$.

Here $\hat{H} = \frac{\hat{p}^2}{2m} + V_0$ so

$$[\hat{H}, \hat{p}] = \left[\frac{\hat{p}^2}{2m} + V_0, \hat{p} \right] = \frac{1}{2m} [\hat{p}^2, \hat{p}] + [V_0, \hat{p}] = 0 \checkmark$$

Thus they can be measured at the same time to arbitrary precision.

Problem 1 continued:

(d) [4 points] Evaluate the commutator $[f(\hat{x}), \hat{p}]$ where $f(\hat{x})$ is an arbitrary differentiable function.

$$\begin{aligned} [f, \hat{p}]g(x) &= f\hat{p}g - \hat{p}fg = f \frac{\hbar}{i} \frac{d}{dx}g - \frac{\hbar}{i} \frac{d}{dx}(fg) \\ &= \frac{\hbar}{i} \left(f \frac{dg}{dx} - \frac{d}{dx}(fg) - g \frac{df}{dx} \right) = - \left(\frac{\hbar}{i} \frac{df}{dx} \right) g. \end{aligned}$$

Thus
$$[f, \hat{p}] = i\hbar \frac{df}{dx}$$

(e) [8 points] Consider the operator

$$\hat{A} = i\hbar \left[(\hat{x}^2 + 1) \frac{d}{dx} + \hat{x} \right].$$

Is this operator Hermitian? [Hint: First write the operator in terms of well known operators that are Hermitian. The solution to the previous problem or a recent homework problem (given in the book) will speed up your justification.]

Using $\hat{p} = -i\hbar \frac{d}{dx}$ we can write.

$$\hat{A} = -(\hat{x}^2 + 1)\hat{p} + i\hbar \hat{x}$$

$$\begin{aligned} \text{So } \hat{A}^\dagger &= -[(\hat{x}^2 + 1)\hat{p}]^\dagger + [i\hbar \hat{x}]^\dagger = -\hat{p}(\hat{x}^2 + 1) - i\hbar \hat{x} \\ &= -\hat{p}\hat{x}^2 - \hat{p} - i\hbar \hat{x}. \end{aligned}$$

since \hat{p}, \hat{x} are Hermitian

Note Recall that $[\hat{x}^2, \hat{p}] = i\hbar \frac{d}{dx}(\hat{x}^2) = 2i\hbar \hat{x}$. from part (d).

$$\text{So } -\hat{p}\hat{x}^2 = 2i\hbar \hat{x} - \hat{x}^2 \hat{p}$$

$$\text{So } \hat{A}^\dagger = \cancel{2i\hbar \hat{x}} - \hat{x}^2 \hat{p} - \hat{p} - \cancel{i\hbar \hat{x}} = -(\hat{x}^2 + 1)\hat{p} + i\hbar \hat{x} = \hat{A}.$$

Thus \hat{A} is Hermitian.

The clever way to do this is see that

$$\hat{A} = -f(\hat{x})\hat{p} + \frac{1}{2} [f(\hat{x}), \hat{p}] \quad \text{where } f(\hat{x}) \equiv \hat{x}^2 + 1$$

Then
$$\hat{A} = -\frac{1}{2} (f(\hat{x})\hat{p} + \hat{p}f(\hat{x})) \quad \text{which is Hermitian.}$$

Solutions

Problem 2. (30 points)

Let the states $\{|\phi_1\rangle, |\phi_2\rangle\}$ be the determinant states of the Hermitian operator \hat{B} with eigenvalues $\{b_1, b_2\}$. Consider the operator

$$\hat{C} = |\phi_1\rangle\langle\phi_1| - i\sqrt{3}|\phi_1\rangle\langle\phi_2| + i\sqrt{3}|\phi_2\rangle\langle\phi_1| - |\phi_2\rangle\langle\phi_2|.$$

[Note: In the answers to the following different representations can be used. Any representation is fine as long as the answers are clear.] There are questions on the back of this page.

(a) [4 points] Is \hat{C} a Hermitian operator?

This is easiest to answer if we write \hat{C} as a matrix in the $\{|\phi_1\rangle, |\phi_2\rangle\}$ basis.

$$C = \begin{pmatrix} 1 & -i\sqrt{3} \\ i\sqrt{3} & -1 \end{pmatrix}. \quad \text{Then } C^\dagger = \begin{pmatrix} 1 & (i\sqrt{3})^* \\ (-i\sqrt{3})^* & -1 \end{pmatrix} = \begin{pmatrix} 1 & -i\sqrt{3} \\ i\sqrt{3} & -1 \end{pmatrix} = C$$

Thus C is Hermitian

(b) [10 points] Find the eigenvalues and normalized eigenvectors for \hat{C} .

$$(C - \lambda I) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -i\sqrt{3} \\ i\sqrt{3} & -1-\lambda \end{vmatrix} = 0 = -(1-\lambda^2) - 3$$

$$\Rightarrow \lambda^2 = 4 \Rightarrow \boxed{\lambda_{\pm} = \pm 2} \leftarrow \text{eigen values.}$$

Let $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$. Then $C|\psi\rangle = \lambda|\psi\rangle$

$$\Rightarrow \begin{pmatrix} 1 & -i\sqrt{3} \\ i\sqrt{3} & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a - i\sqrt{3}b \\ i\sqrt{3}a - b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\lambda = 2: \quad i\sqrt{3}a - b = 2b \Rightarrow 3b = i\sqrt{3}a \Rightarrow b = \frac{i}{\sqrt{3}}a.$$

$$\text{Normalizing: } |a|^2 + |b|^2 = 1 \Rightarrow |a|^2 \left(1 + \frac{1}{3}\right) = 1 \Rightarrow a = \frac{\sqrt{3}}{2}.$$

$$\text{So } \lambda_+ = 2; \quad |\psi_+\rangle = \frac{1}{2} \begin{pmatrix} \sqrt{3} \\ i \end{pmatrix}$$

$$\lambda = -2: \quad i\sqrt{3}a - b = -2b \Rightarrow b = -i\sqrt{3}a.$$

$$\text{So } \lambda_- = -2; \quad |\psi_-\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ -i\sqrt{3} \end{pmatrix}$$

Recall that these vectors are only defined up to a phase. Other forms are equivalent

Problem 2 continued:

- (c) [2 points] Suppose the system starts in the state $|\mathcal{S}\rangle = |\phi_2\rangle$. What is the probability that a measurement of \hat{B} will give the value b_1 ?

0% Since $|\phi_2\rangle$ is a determinant state of \hat{B}
then every measurement of \hat{B} gives b_2 .

- (d) [6 points] For the initial state $|\mathcal{S}\rangle = |\phi_2\rangle$ calculate $\langle \hat{C} \rangle$.

$$\langle \hat{C} \rangle = \langle \phi_2 | \hat{C} | \phi_2 \rangle = C_{22} \leftarrow \text{matrix element}$$

$$\Rightarrow \langle \hat{C} \rangle = -1$$

Alternatively as vectors and matrices

$$\langle \hat{C} \rangle = (0 \ 1) \begin{pmatrix} 1 & -i\sqrt{3} \\ i\sqrt{3} & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (0 \ 1) \begin{pmatrix} -i\sqrt{3} \\ -1 \end{pmatrix} = -1 \checkmark$$

- (e) [8 points] For the initial state $|\mathcal{S}\rangle = |\phi_2\rangle$ suppose we measure \hat{C} . What values can we get and with what probabilities? What is the sum of these probabilities? What should the sum be?

We must measure one of the eigenvalues; $\lambda_{\pm} = \pm 2$.

The probability is $P_{\pm} = |C_{\pm}|^2$ where $C_{\pm} = \langle \psi_{\pm} | \mathcal{S} \rangle$.

$$C_{+} = \langle \psi_{+} | \phi_2 \rangle = \frac{1}{2} (\sqrt{3} - i) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -i/2$$

$$C_{-} = \langle \psi_{-} | \phi_2 \rangle = \frac{1}{2} (1 + i\sqrt{3}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{i\sqrt{3}}{2}$$

Thus we measure

$$\lambda_{+} = +2 \text{ with } P_{+} = \frac{1}{4}$$
$$\lambda_{-} = -2 \text{ with } P_{-} = \frac{3}{4}$$

Notice $P_{+} + P_{-} = 1$ as it must.