Solutions

Problem 1. (24 points)

Clearly explain your reasoning. **There are questions on the back of this page.**

(a) [6 points] Explain why we say there is no such thing as a classical free particle in quantum mechanics. How do we describe a "free particle" in quantum mechanics?

In Classical mechanics a free particle has a definite energy E.
In QM such a state is <u>NOT</u> normalizable, thus it cannot ~~def~~ describe a physical state. Instead we use a wave packet: a linear combination of single energy states.

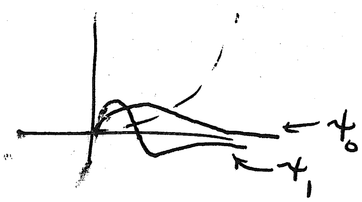
A free particle in QM does not have a specific energy.

(b) [6 points] For the half harmonic oscillator potential

$$V(x) = \begin{cases} \infty, & x \le 0 \\ \frac{1}{2} m\omega^2 x^2, & x > 0 \end{cases}$$

what are the energies of the two lowest states? Express your answer in terms of m, ω, and constants, as appropriate.

The wave functions of the half harmonic oscillator are just like the normal harmonic oscillator states, except, $\psi \to 0$ at $x = 0$. Thus we can match them to harmonic oscillator states with 1 and 3 nodes.



Thus
$$\boxed{\begin{aligned} E_0 &= E_1^{SHO} = \left(1+\tfrac{1}{2}\right)\hbar\omega = \tfrac{3}{2}\hbar\omega \\ E_1 &= E_3^{SHO} = \tfrac{7}{2}\hbar\omega \end{aligned}}$$

Problem 1 continued:

(c) [4 points] Let $\{\psi_n\}$ be the set of wavefunctions that are solutions to the harmonic oscillator potential with frequency ω for a particle of mass m. Consider the wavefunction

$$\psi(x) = A\,(3\psi_1 + 4\psi_3)\,.$$

Find A.

$$1 = \int |\psi|^2 dx = |A|^2 \int 9|\psi_1|^2 + 16|\psi_3|^2 + 12(\underbrace{\psi_1^*\psi_3}_{0} + \underbrace{\psi_1\psi_3^*}_{0})\,dx$$

$$= |A|^2\,(9+16)$$

$$\Rightarrow \boxed{A = \frac{1}{5}}$$

(d) [4 points] For the wave function in the previous part what is $\langle \hat{x} \rangle$? [*Hint*: The answer can be calculated without doing any integrals. One way to do this is to use the symmetries of the wave functions.]

$$\langle \hat{x} \rangle = \frac{1}{25}\int (3\psi_1^* + 4\psi_3^*)\,x\,(3\psi_1 + 4\psi_3)\,dx$$

$$= \frac{1}{25}\int_{-\infty}^{\infty} 9\,\psi_1^* x\,\psi_1 + 16\,\psi_3^* x\,\psi_3 + 12\,(\psi_1^* x \psi_3 + \psi_3^* x \psi_1)\,dx.$$

But ψ_1 and ψ_3 are odd functions. Thus all the products $\psi^* x \psi$ are odd. Thus all the integrals are 0. Therefore $\boxed{\langle \hat{x} \rangle = 0}$

(e) [4 points] For this wave function what is the expectation value of the energy, $\langle \hat{H} \rangle$? Express your answer in terms of m, ω, and constants, as appropriate.

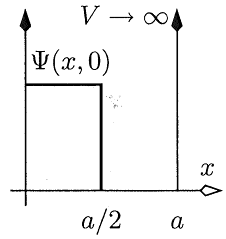
$$\langle \hat{H} \rangle = \frac{1}{25}\left[9E_1 + 16E_3\right] = \frac{1}{25}\hbar\omega\left[9\left(\tfrac{3}{2}\right) + 16\left(\tfrac{7}{2}\right)\right]$$

16
7
112
+27
139

$$\boxed{\langle \hat{H} \rangle = \frac{139}{50}\hbar\omega}$$

Solutions

Problem 2. (20 points)
A particle of mass m is in an infinite square well given by the potential



$$V(x) = \begin{cases} \infty, & x \leq 0 \\ 0, & 0 < x < a \\ \infty, & x \geq a \end{cases} .$$

The particle is placed in the left half of the well with the initial state

$$\Psi(x,0) = \begin{cases} 0, & x \leq 0 \\ A, & 0 < x < a/2 \\ 0, & x \geq a/2 \end{cases} .$$

(a) [4 points] Calculate the value of A.

$$1 = \int |\Psi(x,0)|^2 dx = |A|^2 \int_0^{a/2} dx = |A|^2 \frac{a}{2} \Rightarrow \boxed{A = \sqrt{\frac{2}{a}}}$$

(b) [10 points] Expand the initial state in terms of the solutions to the time independent Schrödinger equation, $\psi_n(x)$. [*Note*: Determining the expansion coefficients is sufficient.]

$$C_n = \int \Psi(x,0)\, \psi_n^*(x)\, dx = \sqrt{\frac{2}{a}} \sqrt{\frac{2}{a}} \int_0^{a/2} \sin \frac{n\pi}{a} x \, dx \; ; n = 1, 2, \ldots$$

$$= \frac{2}{a} \left(\frac{a}{n\pi}\right) \left(- \cos \frac{n\pi x}{a}\right)\Big|_0^{a/2} = \underline{\frac{2}{n\pi}\left(1 - \cos \frac{n\pi}{2}\right)}$$

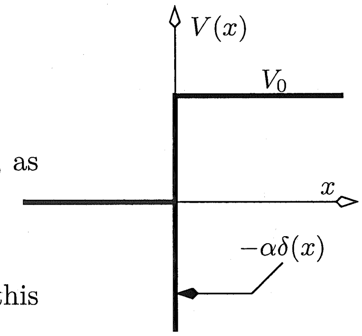
So $\boxed{C_n = \begin{cases} \frac{2}{n\pi}, & n \text{ odd} \\ \frac{4}{n\pi}, & n = 2, 6, 10, \ldots \\ 0, & n = 4, 8, \ldots \end{cases}}$

(c) [6 points] Write down the energy of the ground state, and first, second, and third exited states. Also write down the probability of finding the particle in each of these states.

$E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}$

n	1	2	3	4
C_n	$2/\pi$	$2/\pi$	$\frac{2}{3\pi}$	0
E_n	$\frac{\pi^2 \hbar^2}{2ma^2}$	$\frac{4\hbar^2\pi^2}{2ma^2}$	$\frac{9\hbar^2\pi^2}{2ma^2}$	$\frac{16\hbar^2\pi^2}{2ma^2}$
prob	$4/\pi^2$	$4/\pi^2$	$4/9\pi^2$	0

Problem 3. (26 points)
Consider the potential given by



$$V(x) = -\alpha\delta(x) + V_0\,\theta(x)$$

where $\theta(x)$ is the step function (it is 1 when $x \geq 0$, zero otherwise), as shown at thet right.
[*Note*: **There are questions on the back of this page.**]

(a) [2 points] What condition must be satisfied by the energy for this potential to have bound states?

$\cancel{E>0}$ $\quad \boxed{E<0}$ $\quad V(x\to-\infty)=0$, $\quad V(x\to\infty)=V_0$

(b) [5 points] Write down the bound state wavefunctions in the region $x < 0$ and $x > 0$. Clearly define any constants you introduce.

$$\boxed{\begin{aligned} x<0:&\quad \psi_I = A e^{\kappa x} \\ x>0:&\quad \psi_{II} = B e^{-\kappa' x}\end{aligned}}$$

$$\kappa \equiv \frac{\sqrt{-2mE}}{\hbar} \qquad \kappa' \equiv \frac{\sqrt{2m(V_0-E)}}{\hbar}$$

(c) [5 points] Use the continuity of the wave function to solve for the wave function up to an overall normalization factor.

$$\psi_I(0) = \psi_{II}(0) \Rightarrow A = B$$

Thus
$$\boxed{\psi(x) = A\begin{cases} e^{\kappa x}, & x<0 \\ e^{-\kappa' x}, & x>0\end{cases}}$$

(d) [5 points] Without doing a calculation do we expect $\langle\hat{x}\rangle < 0$, $\langle\hat{x}\rangle = 0$, or $\langle\hat{x}\rangle > 0$ for the bound state? Justify your answer.

$\kappa < \kappa'$ thus ψ decays more quickly for $x>0$

Thus expect $\boxed{\langle\hat{x}\rangle < 0}$

Problem 3 continued:

(e) [6 points] Find an equation the determines the energy of the bound state in terms of m, α, E, V_0, and constants, as necessary. Do **not** solve the equation, leave it in a clear form.

$|V| \to \infty$ at $x=0$, thus

$$\Delta\left(\frac{d\psi}{dx}\right)\Big|_{0^-}^{0^+} = \frac{2m}{\hbar^2}\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} V(x)\,\psi(x)\,dx$$

$$\Rightarrow \quad -k' - k = -\frac{2m}{\hbar^2}\alpha$$

$$\Rightarrow \quad \frac{\sqrt{2m(V_0-E)}}{\hbar} + \frac{\sqrt{-2mE}}{\hbar} = \frac{2m}{\hbar^2}\alpha$$

$$\Rightarrow \quad \boxed{\sqrt{V_0-E} + \sqrt{-E} = \frac{\sqrt{2m}}{\hbar}\alpha} \qquad (*)$$

(f) [3 points] How many bound states are there? Solve for the energy of the bound states.

Squaring $(*)$ $\Rightarrow$ $V_0 - E - E + 2\sqrt{E^2 - EV_0} = \frac{2m}{\hbar^2}\alpha^2$

$$\Rightarrow \quad 2\sqrt{E^2 - EV_0} = \frac{2m}{\hbar^2}\alpha^2 + 2E - V_0$$

Squaring again $\Rightarrow$ $4(\cancel{E^2} - \cancel{EV_0}) = \cancel{4E^2} - \cancel{4EV_0} + \left(\frac{2m}{\hbar^2}\alpha^2\right)^2 + V_0^2 + \frac{8m}{\hbar^2}\alpha^2 E - \frac{4m}{\hbar^2}\alpha^2 V_0$

$$\Rightarrow \quad \cancel{4}\left(\frac{2m\cancel{\alpha^2}}{\cancel{\hbar^2}}\right)E = -\frac{1}{4}\left(\frac{2m\alpha^2}{\hbar^2}\right)^{\cancel{2}} - \frac{V_0^2}{4(2m\alpha^2/\hbar^2)} + \cancel{2}\cancel{\left(\frac{2m\alpha^2}{\hbar^2}\right)}\frac{V_0}{2}$$

$$\Rightarrow \quad \boxed{E = -\frac{m\alpha^2}{2\hbar^2} - \frac{\hbar^2}{8m\alpha^2}V_0^2 + \frac{1}{2}V_0}$$

There is 1 bound state

Notice if $V_0 = 0$ we get the same energy as in eq (2.129) in Griffiths.