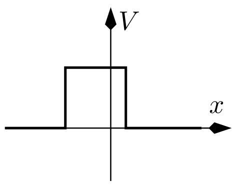
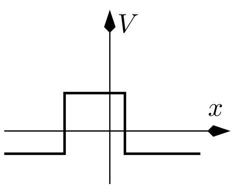
Problem 1. (28 points)

[***Note*: There are questions on the back of this page.**] For each of the following potentials state whether they can have quantum mechanical bound states or not. Justify your answers. [12 points]

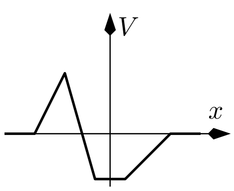
(a)



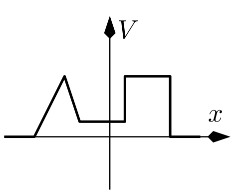
(b)



(c)

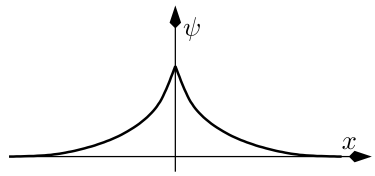


(d)

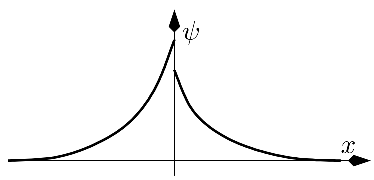


Problem 1 continued:
For each of the following functions state whether they could be quantum mechanical wave functions or not. Justify your answers. [16 points]

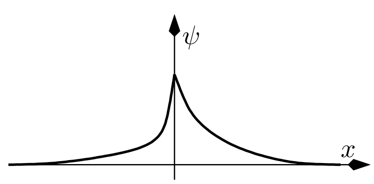
(e)



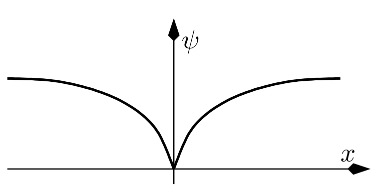
(f)



(g)



(h)



Problem 2. (26 points)

(a) [6 points] A system is in the state

$$|\psi\rangle = \frac{1}{\sqrt{15}}|1\rangle + \frac{1}{\sqrt{3}}|2\rangle + \sqrt{\frac{3}{5}}|3\rangle$$

where $\hat{B}|n\rangle = (n+1)|n\rangle$. Calculate $\langle\hat{B}\rangle$ and $\langle\hat{B}^2\rangle$.

(b) [6 points] Is the operator

$$\hat{Q} = \hat{x}\frac{d}{dx} + 2$$

Hermitian? Justify your answer.

Problem 2 continued:

(c) [4 points] Calculate the matrix representation of the ladder operator $\hat{a}_+$ in the basis of eigenstates for the simple harmonic oscillator Hamiltonian, $\{|n\rangle\}$, where $\hat{H}|n\rangle = E_n|n\rangle$. This is an infinite dimensional matrix so just calculate it up to $n = 3$.

(d) [6 points] For angular momentum $\ell = 2$ find the angle between the total angle momentum, $\boldsymbol{L}$, and its z-component, L_z, for all the allowed projections.

(e) [4 points] For any operator, $\hat{A}$, that is independent of time show that $[\hat{H}, \hat{A}]\Psi(x,t) = 0$. [*Hint*: Both Ψ and $\hat{A}\Psi$ satisfy the time dependent Schrödinger equation.]

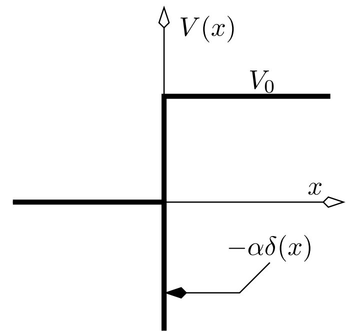
Problem 3. (30 points)

Consider the potential given by

$$V(x) = -\alpha\delta(x) + V_0\,\theta(x)$$

where $\theta(x)$ is the step function (it is 1 when $x \geq 0$, zero otherwise), as shown at the right.

[*Note*: **There are questions on the back of this page.**]



(a) [3 points] For a beam of particles sent in from the left with $0 < E < V_0$ sketch the probability density, $|\psi|^2$.

(b) [4 points] For a beam of particles sent in from the left with $0 < E < V_0$ what is the value of the reflection coefficient? State the answer without calculation and justify your answer.

(c) [3 points] For a beam of particles sent in from the left with $E > V_0$ sketch the probability density, $|\psi|^2$.

Problem 3 continued:

(d) [20 points] Calculate the ratio of the probability density of the transmitted wave function to that of the incident wave function. This is *related* to the inverse of the transmission function, T^{-1}, but isn't quite the same (see problem 2.34 in the book if you are interested, the details aren't relevant for this problem). To check your result evaluate your expression for $V_0 = 0$ and compare it to the known result (2.141) from the book. (When $V_0 = 0$ the ratio **is** the inverse of the transmission function so we can make a direct comparison.)

Problem 4. (30 points)

A particle with mass M is constrained to move on the surface of a sphere with radius R. It can move freely on the sphere but not off the sphere. The Hamiltonian for this system is

$$\hat{H} = \frac{\hat{L}^2}{2MR^2}.$$

[***Note*: There are questions on the back of this page.**]

(a) [6 points] Write down the eigenvalues and eigenstates for this Hamiltonian. What is the degeneracy of the eigenvalues? [*Hint*: You can just "write down" the answers with appropriate explanations.]

(b) [9 points] Suppose the particle starts in the state

$$\psi(\boldsymbol{r}) = \frac{1}{\sqrt{4\pi}} \left\{ \frac{3}{\sqrt{5}} \cos\theta + \frac{1}{\sqrt{12}} \left(1 - 3\cos^2\theta\right) - i\sqrt{5} \sin\theta \cos\theta \sin\phi \right\}.$$

Write this state in terms of the eigenstates of the Hamiltonian.

Problem 4 continued:

(c) [5 points] Suppose we measure the energy for the state in part (b), what values can we get and with what probabilities?

(d) [5 points] Suppose we measure the square of the angular momentum for the state in part (b), what values can we get and with what probabilities?

(e) [5 points] Suppose we measure the z-component of the angular momentum for the state in part (b), what values can we get and with what probabilities?

Problem 5. (23 points)

The mathematical formulation of quantum mechanics we have developed was first formalized by Dirac and is known as "Hermitian quantum mechanics" since Hermitian operators play a central role, in particular they serve as observables since they are guaranteed to have real eigenvalues. It turns out that being Hermitian is a sufficient, but not necessary condition. An alternative formulation is known as "PT quantum mechanics" which constructs a theory symmetric under parity (P) and time-reversal (T). We will study a two state system given by an operator, $\hat{Q}$, with a matrix representation

$$Q = \begin{pmatrix} 13 & -5i \\ -5i & -13 \end{pmatrix}.$$

There are questions on the back of this page.]

(a) [2 points] Show that $\hat{Q}$ is not Hermitian.

(b) [5 points] Despite the previous part, calculate the eigenvalues of $\hat{Q}$ and see that they are real.

Problem 5 continued:

(c) [8 points] Calculate the normalized eigenvectors, $\{|e_+\rangle, |e_-\rangle\}$, for $\hat{Q}$.

(d) [4 points] Calculate the inner product $\langle e_+|e_-\rangle$ and see that the eigenvectors are *not* orthogonal (as they would be for a Hermitian operator).

(e) [4 points] The reason for the previous part is that we used the wrong inner product. The PT inner product is defined without the complex conjugation as

$$\langle\alpha|\beta\rangle_{\mathrm{PT}} \equiv |\alpha\rangle^{\mathsf{T}} |\beta\rangle = \sum_j \alpha_j \beta_j.$$

With this inner product show that the eigenvectors we found in (c) are orthogonal.

Problem 6. (28 points)

Three non-interacting particles each of mass m are placed in an infinite square well potential of width a. For the following questions write the energies in terms of m, a, and constants, as appropriate. When asked for the degeneracy of the states this means the degeneracy of the three particle state, that is, how many ways are there to arrange the three particles. Make sure you explain your answers, a sketch may help. [*Note*: **There are questions on the back of this page.**]

(a) [4 points] If the particles are spinless and distinguishable what is the energy of the ground state of the system and what is its degeneracy?

(b) [4 points] If the particles are spinless and distinguishable what is the energy of the first excited state of the system and what is its degeneracy?

(c) [4 points] If the particles are identical and spin-0 what is the energy of the ground state of the system and what is its degeneracy?

Problem 6 continued:

(d) [4 points] If the particles are identical and spin-0 what is the energy of the first excited state of the system and what is its degeneracy?

(e) [6 points] If the particles are identical and spin-1/2 what is the energy of the ground state of the system and what is its degeneracy? (Do not ignore the spin of the particles!)

(f) [6 points] If the particles are identical and spin-1/2 what is the energy of the first excited state of the system and what is its degeneracy? (Do not ignore the spin of the particles!)