Solutions

**Problem 1.** (20 points)

**There are questions on the back of this page.**

(a) An electron is placed at rest in a constant magnetic field, $\mathbf{B} = B_0\hat{k}$, with $B_0 = 1$ T. Calculate the magnitude of the energy difference between the two spin states of the electron in this magnetic field. For the gyromagnetic ratio of the electron use $\gamma = -e/m_e$ where $m_e$ is the mass of the electron [*Note*: The value of the magnetic field here and the values of constants in the book are given in SI units. For this part of the problem it is best to quote the energy in electron-volts. The conversion factor is 1 eV $= 1.6 \times 10^{-19}$ J.]

$$E_{\pm} = \pm \frac{\gamma B_0 \hbar}{2}, \quad \gamma = \frac{-e}{m_e}$$

Thus $\boxed{|\Delta E| = \gamma B_0 \hbar = 1.16 \times 10^{-4}\,\text{eV}}$

(b) For the electron in the previous part calculate the precession frequency (Larmor frequency) of the spin.

$$\boxed{\omega = \gamma B_0 = 1.76 \times 10^{11}\ \text{rad/s}}$$

**Problem 1 continued:**

**(c)** The potential energy of a spherical harmonic oscillator of mass $M$ and angular frequency $\omega$ is given by $V(r) = M^2\omega^2 r^2/2$. Write down the differential equation that the radial part of the wavefunction, $R(r)$, must satisfy. [*Note*: It is easier and sufficient to write down the differential equation that $u(r) \equiv rR(r)$ must satisfy. Do **not** try to solve this differential equation!]

From eq [4.37]

$$\boxed{-\frac{\hbar^2}{2M}\frac{d^2u}{dr^2} + \left[\frac{1}{2}M\omega^2 r^2 + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu}$$

**(d)** What are the solutions to the angular part of Schrödinger's equation for the spherical harmonic oscillator?

Since $V(r)$ the angular solutions are

the Spherical Harmonics $Y_l^m(\theta, \varphi)$

Solutions.

**Problem 2.** (25 points)

At time $t = 0$ a hydrogen atom is in the superposition state

$$\Psi(\mathbf{r}, 0) = \frac{4}{(2a_0)^{3/2}} \left[ e^{-r/a_0} + A\frac{r}{a_0} e^{-r/2a_0} \left( -iY_1^1 + Y_1^{-1} + \sqrt{7} Y_1^0 \right) \right],$$

where $A$ is a constant and $a_0$ is the Bohr radius.

**(a)** [6 points] Rewrite this wave function in terms of the eigenstates of the Hamiltonian, $\psi_{n\ell m}(\mathbf{r})$.

$R_{10}(r) = \frac{2}{a_0^{3/2}} e^{-r/a_0}$, $R_{21}(r) = \frac{1}{\sqrt{24}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0}$, $Y_0^0 = \frac{1}{\sqrt{4\pi}}$

Thus

$$\Psi(\vec{r}, 0) = \frac{4}{2\sqrt{2}\, a_0^{3/2}} \left[ \frac{\sqrt{4\pi}\, a_0^{3/2}}{2} \psi_{100} + A\sqrt{24}\, a_0^{3/2} \left( -i\psi_{211} + \psi_{2,1,-1} + \sqrt{7}\psi_{210} \right) \right]$$

$$\Rightarrow \boxed{\Psi(\vec{r}, 0) = \frac{1}{\sqrt{2}} \left[ \sqrt{4\pi}\, \psi_{100} + A\, 4\sqrt{6} \left( -i\psi_{211} + \psi_{21,-1} + \sqrt{7}\psi_{210} \right) \right]}$$

Note: This makes no sense! $P(\psi_{100}) > 1$. There should have been a $Y^0$ in the first term.

**(b)** [9 points] Find the constant $A$. [*Note*: Because of the previous part you do **not** need to evaluate any integrals!]

Since $\Psi(\vec{r}, 0)$ is expanded in eigenstates of the Hamiltonian (orthonormal states) the sum of the coefficients squared must be 1: Thus

$$\frac{1}{2}\left( 4\pi + |A|^2 \cdot 16 \cdot 6 \cdot (1+1+7) \right) = 1$$

$$\Rightarrow |A|^2 \cdot 16 \cdot 9 \cdot 3 = 1 - 2\pi$$

$$\Rightarrow A = \frac{\sqrt{1-2\pi}}{4 \cdot 3 \cdot \sqrt{3}} \Rightarrow \boxed{A = \frac{i}{12}\sqrt{\frac{2\pi - 1}{3}}}$$

$$A \cdot 4\sqrt{6} = \frac{i}{\sqrt{2}}\sqrt{\frac{2\pi - 1}{3}} \quad \overset{3}{\cancel{4}}\sqrt{\overset{\sqrt{2}}{\cancel{6}}}$$

**Problem 2 continued:**

**(c)** [5 points] For the given initial state what is $\Psi(\mathbf{r}, t)$.

$$\boxed{\Psi(\vec{r},t) = \frac{1}{\sqrt{2}}\left\{\sqrt{4\pi}\,\psi_{100}\,e^{-iE_1 t/\hbar} + \frac{i}{\sqrt{3}}\sqrt{4\pi - 2}\left(-i\psi_{211} + \psi_{21,-1} + \sqrt{7}\,\psi_{210}\right)\right\}}$$

$$\times e^{-iE_2 t/\hbar}$$

where $E_1 = -13.6\,\text{eV}$

$E_2 = -3.4\,\text{eV}$

**(d)** [5 points] If at $t = 0$ a measurement of $\hat{L}_z$ is made and we find the value $-\hbar$, what will $\Psi(\mathbf{r}, t)$ be now?

If we measure $m_\ell = -1$ for $\hat{L}_z$ we are left in the state

$$\Psi(\vec{r}, 0) = \psi_{21,-1}$$

Thus

$$\boxed{\Psi(\vec{r}, t) = \psi_{21,-1}\,e^{-iE_2 t/\hbar}}$$

Solutions

**Problem 3.** (25 points)

An electron is in the angular momentum state

$$\psi = \frac{1}{\sqrt{2}}\left(Y_2^1\chi_- - Y_2^{-1}\chi_+\right),$$

where $\chi_\pm$ are the spin eigenstates. [*Note*: **There are questions on the back of this page.**]

**(a)** [5 points] What are $\langle \hat{L}_x \rangle$ and $\langle \hat{L}_y \rangle$ for this state?

$\hat{L}_x = \frac{1}{2}(\hat{L}_+ + \hat{L}_-)$. Thus $\langle Y_2^{\pm 1} | \hat{L}_x | Y_2^{\pm 1} \rangle = 0$

and $\langle Y_2^{\pm 1} | \hat{L}_x Y_2^{\mp 1} \rangle = 0$

So $\boxed{\langle \hat{L}_x \rangle = 0}$

By a similar argument $\boxed{\langle \hat{L}_y \rangle = 0}$

**(b)** [7 points] The spin-orbit coupling, $\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$, is an important contribution to the spectrum of the hydrogen atom that breaks some of the degeneracy in the energy of states. Here calculate the expectation value $\langle \hat{\mathbf{L}} \cdot \hat{\mathbf{S}} \rangle$ for this state.

$$\langle \hat{\vec{L}} \cdot \hat{\vec{S}} \rangle = \langle \hat{L}_x \hat{S}_x + \hat{L}_y \hat{S}_y + \hat{L}_z \hat{S}_z \rangle$$

From (a) this simplifies to

$$\langle \hat{\vec{L}} \cdot \hat{\vec{S}} \rangle = \langle \hat{L}_z \hat{S}_z \rangle = \frac{1}{2}\left(\hbar\left(-\frac{\hbar}{2}\right) + \cancel{(-\hbar)\left(\frac{\hbar}{2}\right)}\right)$$

$$\Rightarrow \boxed{\langle \hat{\vec{L}} \cdot \hat{\vec{S}} \rangle = -\frac{\hbar^2}{2}}$$

**Problem 3 continued:**

**(c)** [3 points] What are the possible values for the $z$-component total angular momentum quantum number, $m_j$, and for the total angular momentum quantum number, $j$, for the given electron state.

We have the states $|2 1\rangle|\tfrac{1}{2}, -\tfrac{1}{2}\rangle$ and $|2,-1\rangle|\tfrac{1}{2}, \tfrac{1}{2}\rangle$

Thus $\boxed{m_j = \pm\tfrac{1}{2}}$ and $j = |2-\tfrac{1}{2}|, \ldots, |2+\tfrac{1}{2}|$

so $\boxed{j = \tfrac{3}{2}, \ 5/2}$

**(d)** [7 points] Write the original state in terms of the basis $\{j, m_j\}$.

For the two states in (c) we use the $2\times\tfrac{1}{2}$ table of Clebsh-Gordon coeff

$$|2 1\rangle|\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{5}}\,|\tfrac{5}{2}, \tfrac{1}{2}\rangle + \sqrt{\tfrac{3}{5}}\,|\tfrac{3}{2}, \tfrac{1}{2}\rangle$$

$$|2,-1\rangle|\tfrac{1}{2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{5}}\,|\tfrac{5}{2}, -\tfrac{1}{2}\rangle - \sqrt{\tfrac{3}{5}}\,|\tfrac{3}{2}, -\tfrac{1}{2}\rangle$$

Thus

$$\boxed{\psi = \frac{1}{\sqrt{2}}\left(\sqrt{\tfrac{2}{5}}\,|\tfrac{5}{2}, \tfrac{1}{2}\rangle + \sqrt{\tfrac{3}{5}}\,|\tfrac{3}{2}, \tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{5}}\,|\tfrac{5}{2}, -\tfrac{1}{2}\rangle + \sqrt{\tfrac{3}{5}}\,|\tfrac{3}{2}, -\tfrac{1}{2}\rangle\right)}$$

**(e)** [3 points] With what probability would each of the values of $j$ from part (c) be measured?

$$\boxed{\begin{aligned} j = \tfrac{5}{2} &: \ P = \tfrac{1}{2}\left(\tfrac{2}{5} + \tfrac{2}{5}\right) = \tfrac{2}{5} \\ j = \tfrac{3}{2} &: \ P = \tfrac{1}{2}\left(\tfrac{3}{5} + \tfrac{3}{5}\right) = \tfrac{3}{5} \end{aligned}}$$