Solutions

Problem 1. (25 points)

There are questions on the back of this page.

(a) [4 points] Consider the displacement operator, $\hat{D}_{x_0}$, that displaces a function along the x-axis by a constant distance x_0. That is, for any function, $\hat{D}_{x_0} f(x) = f(x + x_0)$. Is the displacement operator Hermitian? Justify your answer.

$$\langle f | \hat{D}_{x_0} g \rangle = \int f^*(x)\, g(x+x_0)\, dx$$

$$\langle \hat{D}_{x_0} f | g \rangle = \int f^*(x+x_0)\, g(x)\, dx$$

since $\langle f | \hat{D}_{x_0} g \rangle \neq \langle \hat{D}_{x_0} f | g \rangle$ $\boxed{\hat{D}_{x_0} \text{ is NOT Hermitian}}$

(b) [6 points] We can write the (unnormalized) eigenfunctions for the displacement operator in the form

$$h_\beta(x) = e^{\beta x} g(x)$$

where β is any complex number and $g(x)$ is periodic, that is, $g(x + x_0) = g(x)$. Find the eigenvalues for this operator in terms of β, x_0, and constants, as appropriate. Is the spectrum discrete or continuous?

Eigenvalue problem: $\hat{D}_{x_0} h_\beta(x) = \lambda_\beta h_\beta(x)$

$$\Rightarrow h_\beta(x+x_0) = e^{\beta(x+x_0)} g(x+x_0) = e^{\beta x_0}\left(e^{\beta x} g(x)\right) = e^{\beta x_0} h_\beta(x)$$

$$= \lambda_\beta h_\beta(x).$$

Thus $\boxed{\lambda_\beta = e^{\beta x_0}}$ since $\beta \in \mathbb{C}$ this is a <u>continuous</u> spectrum.

Problem 1 continued:

(c) [5 points] For two Hermitian operators $\hat{A}$ and $\hat{B}$ show that their observations are compatible, that is we can have $\sigma_A \sigma_B = 0$, if the product of these two operators, $\hat{A}\hat{B}$, is also Hermitian.

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A},\hat{B}] \rangle\right)^2.$$

Since $(\hat{A}\hat{B})^\dagger = \hat{A}\hat{B}$ we see that $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger \hat{A}^\dagger = \hat{B}\hat{A} = \hat{A}\hat{B}$.

Thus $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = 0$.

Thus $\underline{\sigma_A \sigma_B \geq 0}$ and the observations are compatible.

(d) [5 points] Show that $[\hat{x}^2, \hat{p}^2] = 2i\hbar(\hat{x}\hat{p} + \hat{p}\hat{x})$.

This can be done in a number of ways. Here we use a direct approach.

$$[\hat{x}^2,\hat{p}^2]f = x^2\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 f - \left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 x^2 f = \hbar^2\left(-x^2 f'' + \frac{d^2}{dx^2}x^2 f\right)$$

$$= \hbar^2\left(-x^2 f'' + \frac{d}{dx}(2xf + x^2 f')\right) = \hbar^2\left(\cancel{-x^2 f''} + 2f + 4xf' + \cancel{x^2 f''}\right)$$

$$= 2\hbar^2\left(1 + 2x\left(\frac{i}{\hbar}\frac{d}{dx}\right)\right)f = 2\left(\hbar^2 + 2i\hat{x}\hat{p}\right)f.$$

Thus $[\hat{x}^2,\hat{p}^2] = 2(\hbar^2 + 2i\hbar\hat{x}\hat{p})$. To get this in the correct form use $[\hat{x},\hat{p}] = i\hbar$.

Then $[\hat{x}^2,\hat{p}^2] = 2(\cancel{\hbar^2} + i\hbar\hat{x}\hat{p} + i\hbar(\hat{p}\hat{x} + \cancel{i\hbar}))$

So $\underline{[\hat{x}^2,\hat{p}^2] = 2i\hbar(\hat{x}\hat{p} + \hat{p}\hat{x})}$ ✓

(e) [5 points] For a particle of mass m and angular frequency ω in a harmonic oscillator potential calculate $d\langle\hat{p}^2\rangle/dt$. [*Hint*: Use the results from the previous part.]

$$\frac{d\langle\hat{p}^2\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H},\hat{p}^2]\rangle, \quad \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

So $[\hat{H},\hat{p}^2] = \frac{1}{2}m\omega^2[\hat{x}^2,\hat{p}^2] = \frac{1}{\cancel{2}}m\omega^2(\cancel{2}i\hbar)(\hat{x}\hat{p} + \hat{p}\hat{x})$

So
$$\boxed{\frac{d\langle\hat{p}^2\rangle}{dt} = -m\omega^2\langle\hat{x}\hat{p} + \hat{p}\hat{x}\rangle}$$

Solutions

Problem 2. (25 points)
Consider the matrix

$$H = \begin{pmatrix} h & (1+i)g \\ (1-i)g & h \end{pmatrix}$$

written in some basis $\{|1\rangle, |2\rangle\}$. **There are questions on the back of this page.**

(a) [5 points] For this matrix to represent a two-state Hamiltonian what conditions must be imposed on h and g?

Need $H^\dagger = H \Rightarrow \begin{pmatrix} h^* & (1+i)g^* \\ (1-i)g^* & h^* \end{pmatrix} = \begin{pmatrix} h & (1+i)g \\ (1-i)g & h \end{pmatrix}$

Thus $\boxed{h, g \text{ must be real}}$

(b) [5 points] Given the constraints in the previous part find the eigenvalues for this system.

Need $\det(H - EI) = 0 \Rightarrow \begin{vmatrix} h-E & (1+i)g \\ (1-i)g & h-E \end{vmatrix} = 0$

$\Rightarrow (h-E)^2 - 2g^2 = 0 \quad \Rightarrow \boxed{E_\pm = h \pm \sqrt{2}\, g}$

(c) [10 points] Find the eigenvectors for this system. Also write them in terms of the orignal $\{|1\rangle, |2\rangle\}$ basis.

$H\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = E_\pm \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \Rightarrow \begin{pmatrix} h & (1+i)g \\ (1-i)g & h \end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = (h \pm \sqrt{2}g)\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$

Thus $\cancel{h\alpha} + (1+i)g\beta = (\cancel{h} \pm \sqrt{2}g)\alpha \Rightarrow \frac{1+i}{\sqrt{2}}\beta = \pm\alpha$

We can write the eigenvectors in many ways.

Here is one way $\boxed{|s_\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1+i}{\sqrt{2}} \\ \pm 1 \end{pmatrix}}$

In the original basis we have

$$\boxed{|s_\pm\rangle = \frac{1}{\sqrt{2}}\left[\frac{1+i}{\sqrt{2}}|1\rangle \pm |2\rangle\right]}$$

Problem 2 continued:

(d) [5 points] Suppose the system starts in the state $|\mathcal{S}(0)\rangle = (4|1\rangle + 3|2\rangle)/5$. What are the possible energy values that can be measured and with what probabilities?

We must measure an eigenvalue: $\underline{E_\pm = h \pm \sqrt{2}\, g}$

$$C_\pm = \langle s_\pm | \mathcal{S}(0)\rangle = \frac{1}{5\sqrt{2}}\left(\tfrac{1-i}{\sqrt{2}}, \pm 1\right)\begin{pmatrix}4\\3\end{pmatrix}$$

$$= \frac{1}{5\sqrt{2}}\left(4\frac{(1-i)}{\sqrt{2}} \pm 3\right).$$

Thus Prob is $|C_\pm|^2 = \frac{1}{50}\left(16+9 \pm \frac{12}{\sqrt{2}}(1-\cancel{i}+1+\cancel{i})\right) = \frac{1}{50}\left(25 \pm 12\sqrt{2}\right).$

So we measure $\boxed{h \pm \sqrt{2}\, g}$ with prob $\boxed{|C_\pm|^2 = \frac{1}{2} \pm \frac{6}{25}\sqrt{2}}$

(e) [10 points] Suppose the system starts in the state $|\mathcal{S}(0)\rangle = |2\rangle$. What is the state of the system, $|\mathcal{S}(t)\rangle$, at some time t? Write your answer in the basis $\{|1\rangle, |2\rangle\}$.

$$|\mathcal{S}(0)\rangle = |2\rangle = \cancel{\tfrac{1}{\sqrt{2}}(|1\rangle - |2\rangle)}\ \frac{1}{\sqrt{2}}\left(|s_+\rangle - |s_-\rangle\right).$$

Thus $$|\mathcal{S}(t)\rangle = \frac{1}{\sqrt{2}}\left(|s_+\rangle e^{-iE_+t/\hbar} - |s_-\rangle e^{-iE_-t/\hbar}\right)$$

$$\Rightarrow \boxed{|\mathcal{S}(t)\rangle = \frac{1}{\sqrt{2}} e^{-iht/\hbar}\left(|s_+\rangle e^{-ig\sqrt{2}t/\hbar} - |s_-\rangle e^{+i\sqrt{2}gt/\hbar}\right)}$$

In terms of the original basis we have

$$|\mathcal{S}(t)\rangle = \frac{1}{2} e^{-iht/\hbar}\begin{pmatrix}\frac{1+i}{\sqrt{2}}\left(e^{-i\sqrt{2}gt/\hbar} - e^{i\sqrt{2}gt/\hbar}\right)\\ e^{-ig\sqrt{2}t/\hbar} + e^{i\sqrt{2}gt/\hbar}\end{pmatrix}$$

$$= e^{-iht/\hbar}\begin{pmatrix}-\frac{1+i}{\sqrt{2}}\, i \sin(\sqrt{2}gt/\hbar)\\ \cos(\sqrt{2}gt/\hbar)\end{pmatrix}$$

Thus $$\boxed{|\mathcal{S}(t)\rangle = e^{-iht/\hbar}\begin{pmatrix}\frac{1-i}{\sqrt{2}}\sin(\sqrt{2}gt/\hbar)\\ \cos(\sqrt{2}gt/\hbar)\end{pmatrix}}$$