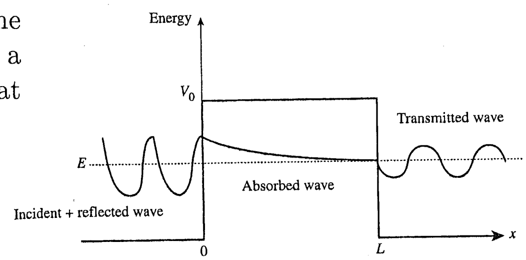
Solutions

Problem 1. (24 points)
Clearly explain your reasoning. **There are questions on the back of this page.**

(a) [4 points] The figure at the right was taken from an introductory quantum mechanics book. The wave sketched in this figure **cannot** represent a wave function. Why not? [*Note*: There is at least one major reason why it cannot.]



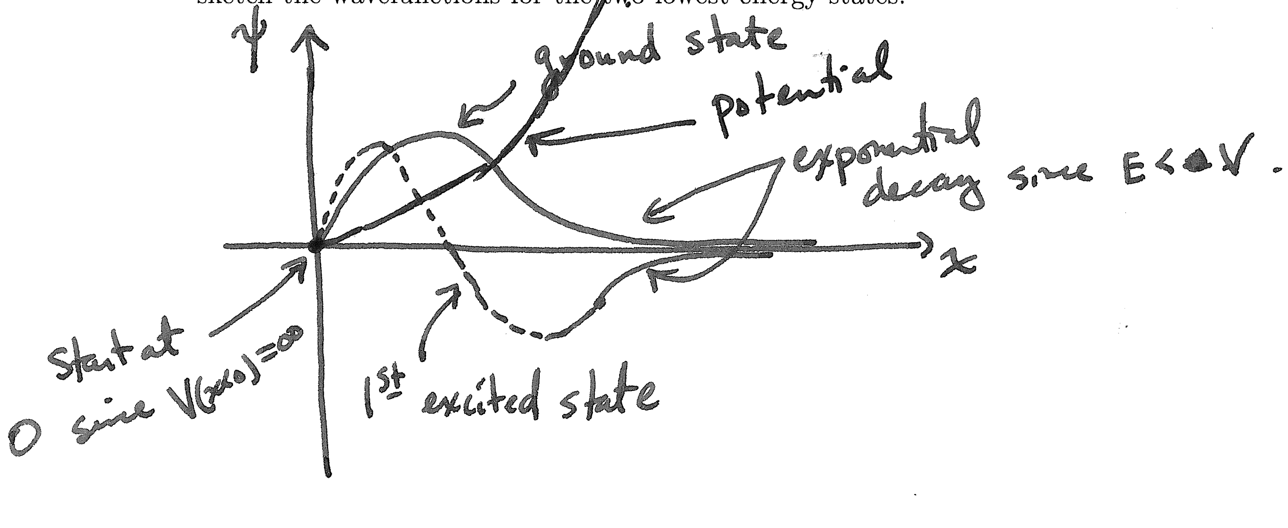
At $x=L$ the derivative is not continuous. Since $V(x)$ is finite the derivative of ψ must be continuous

Note: for $x<0$ $\psi(x)$ does not look the same as for $x>L$. Since the energy of the wave is the same in both regions I would expect ψ to look ~~the same~~ similar, even with interference.

(b) [4 points] For the half harmonic oscillator potential

$$V(x) = \begin{cases} \infty, & x \le 0 \\ \frac{1}{2}m\omega^2x^2, & x > 0 \end{cases}$$

sketch the wavefunctions for the two lowest energy states.



Problem 1 continued:

(c) [4 points] Let $\{\psi_n\}$ be a set of wavefunctions that are solutions to the time independent Schrödinger equation, $\hat{H}\psi_n = E_n\psi_n$. Consider the wavefunction

$$\psi(x) = A\left(3\psi_1 + 2\psi_2 + \psi_3\right).$$

Find A.

$$\int |\psi|^2 dx = 1 = |A|^2 (9+4+1)$$

$$\Rightarrow \boxed{A = \frac{1}{\sqrt{14}}}$$

(d) [8 points] For the wave function in the previous part suppose the energy of the system is measured. What value(s) of the energy would be measured? With what probabilities?

Measured E	Probability
E_1	$9/14$
E_2	$4/14$
E_3	$1/14$

(e) [4 points] For this wave function what is the expectation value of the energy, $\langle \hat{H} \rangle$?

$$\boxed{\langle \hat{H} \rangle = \frac{1}{14}\left(9E_1 + 4E_2 + E_3\right)}$$

Solutions

Problem 2. (20 points)

Consider a particle of mass m, in an infinite square well

$$V(x) = \begin{cases} \infty, & x \le 0 \\ 0, & 0 < x < a \\ \infty, & x \ge a \end{cases}$$

in the state with energy E_n. Answer the following questions **without** doing any integrals. If you feel the need to do an integral then you will be spending way too much time on this problem! Express your answer in terms of a, m, n, and constants, as appropriate. [*Note*: **There are questions on the back of this page.**]

(a) [2 points] Calculate the expectation of the total energy, $\langle \hat{H} \rangle$.

$$\boxed{\langle \hat{H} \rangle = E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}}$$

(b) [2 points] Calculate the expectation of the potential energy, $\langle \hat{V} \rangle$.

$$\boxed{\langle \hat{V} \rangle = 0}$$

(c) [2 points] Calculate the expectation of the kinetic energy, $\langle \hat{T} \rangle$,

$$\boxed{\langle \hat{T} \rangle = \langle \hat{H} \rangle = E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}}$$

(d) [3 points] Calculate the expection of the momentum squared, $\langle \hat{p}^2 \rangle$.

$$\boxed{\langle \hat{p}^2 \rangle = 2m \langle \hat{T} \rangle = \frac{n^2 \pi^2 \hbar^2}{a^2}}$$

(e) [2 points] Calculate the expection of the momentum, $\langle \hat{p} \rangle$.

$\boxed{\langle \hat{p} \rangle = 0}$ [stationary state]

(f) [2 points] Calculate the expection of the position, $\langle \hat{x} \rangle$.

$\boxed{\langle \hat{x} \rangle = a/2}$ [symmetry]

Problem 2 continued:

(g) [4 points] Use the uncertainy principle to calculate an upper limit on the expection of the position squared, $\langle \hat{x}^2 \rangle$.

$$\sigma_x^2 \sigma_p^2 \geq \frac{\hbar^2}{4}$$

$$\Rightarrow \left(\langle \hat{x}^2 \rangle - \frac{a^2}{4}\right)\left(\frac{n^2\pi^2\cancel{\hbar^2}}{a^2}\right) \geq \frac{\cancel{\hbar^2}}{4}$$

$$\Rightarrow \boxed{\langle \hat{x}^2 \rangle \geq \frac{a^2}{4}\left[1 + \frac{1}{n^2\pi^2}\right]}$$

(h) [3 points] For $n = 1$ compare you limit in the previous part to the exact result

$$\langle \hat{x}^2 \rangle = \frac{a^2}{4}\left[\frac{4}{3} - \frac{2}{n^2\pi^2}\right]$$

as one check of your answer.

For $n=1$ we need

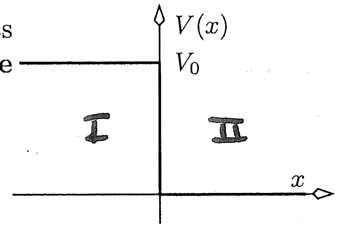
$$\frac{a^2}{4}\left[\frac{4}{3} - \frac{2}{\pi^2}\right] \geq \frac{a^2}{4}\left[1 + \frac{1}{\pi^2}\right]$$

$$\Rightarrow \frac{1}{3} \geq \frac{3}{\pi^2} \Rightarrow 1 \geq \frac{9}{\pi^2} \checkmark$$

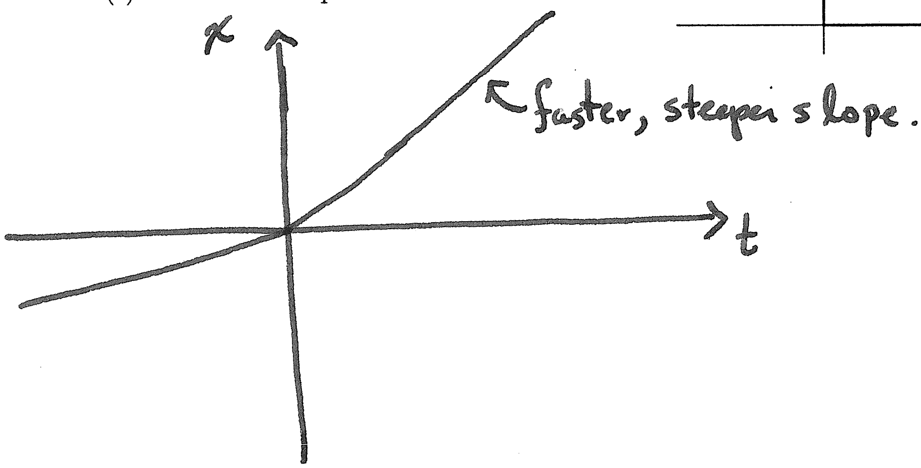
True since $\pi > 3$.

Solutions

Problem 3. (26 points)
For the step potential shown at the right consider a particle of mass m with $E > V_0$ coming from the left (negative x). [*Note*: **There are questions on the back of the page.**]



(a) [4 points] Sketch $x(t)$ for a classical particle.



(b) [10 points] Calculate the reflection coefficient, R, for a quantum mechanical wave. Express your answer in terms of E, V_0, and constants, as appropriate.

$$\psi_I = Ae^{ikx} + Be^{-ikx} \qquad k \equiv \frac{\sqrt{2m(E-V_0)}}{\hbar}$$

$$\psi_{II} = Fe^{ik'x} \qquad k' \equiv \frac{\sqrt{2mE}}{\hbar}$$

BC: ① $\psi_I(0) = \psi_{II}(0) \Rightarrow A + B = F$

② $\psi_I'(0) = \psi_{II}'(0) \Rightarrow k(A-B) = k'F$

$R = \frac{|B|^2}{|A|^2}$ so $k'① - ② \Rightarrow -(k-k') + (k+k')B = 0$

$$\Rightarrow B = \frac{k-k'}{k+k'}A \Rightarrow R = \frac{(k-k')^2}{(k+k')^2}.$$

$$\Rightarrow \boxed{R = \frac{2E - V_0 - 2\sqrt{E(E-V_0)}}{2E - V_0 + 2\sqrt{E(E-V_0)}}}$$

Problem 3 continued:

(c) [5 points] The transmission coefficient, T, is **not** just of the form $|F|^2/|A|^2$. What is the correct expression for the transmission coefficient?

$$\boxed{T = \frac{|\psi_t|^2 v_t}{|\psi_i|^2 v_i} = \frac{|F|^2}{|A|^2}\sqrt{\frac{E}{E-V_0}}}$$

(d) [5 points] Using the previous part calculate the transmission coefficient for a quantum mechanical wave.

From (b) $k(1)+(2) \Rightarrow 2kA = (k+k')F$

So from (c) $T = \frac{4k^2}{(k+k')^2}\sqrt{\frac{E}{E-V_0}} = \frac{4(E-V_0)}{2E-V_0+2\sqrt{E(E-V_0)}}\sqrt{\frac{E}{E-V_0}}$

So $$\boxed{T = \frac{4\sqrt{E(E-V_0)}}{2E-V_0+2\sqrt{E(E-V_0)}}}$$

(e) [2 points] Verify that $R + T = 1$.

From (b) and (d)

$$R+T = \frac{2E-V_0-2\sqrt{E(E-V_0)}+4\sqrt{E(E-V_0)}}{2E-V_0+2\sqrt{E(E-V_0)}} = 1 \checkmark$$