

**Problem 1.** (23 points)

[*Note: There are questions on the back of this page.*]

(a) [4 points] Consider the potential defined by

$$V(x) = \begin{cases} -V_0, & x < -a \\ 0, & -a < x < 0 \\ \infty, & x > 0 \end{cases}.$$

Can this potential have bound states? Justify your answer.

(b) [4 points] For the potential in the previous part a wave is sent in from  $x < 0$  with  $-V_0 < E < 0$ . Sketch  $|\psi|^2$ .

(c) [4 points] For the potential in the previous part a wave is sent in from  $x < 0$  with  $E > 0$ . Sketch  $|\psi|^2$ .

**Problem 1 continued:**

- (d) [8 points] A particle of mass  $m$  in simple harmonic oscillator potential with angular frequency  $\omega$  is in the state

$$\Psi(x, t) = \frac{1}{\sqrt{2}} [\Psi_0(x, t) - \Psi_1(x, t)],$$

where  $\Psi_n(x, t)$  are the solutions to the time dependent Schrödinger equation. Calculate  $\langle \hat{x} \rangle$  for this state in terms of  $m$ ,  $\omega$ , and constants, as appropriate. [Note: No integrals are required nor allowed!]

- (e) [3 points] Calculate  $\langle \hat{p} \rangle$  for the state in the previous part. [Hint: You can do this problem directly using the result of the previous part.]

**Problem 2.** (28 points)

(a) [4 points] A harmonic oscillator has a mass 1 g, frequency 1 Hz, and passes through the equilibrium position with speed 10 cm/s. Estimate the energy quantum number associated with this system.

(b) [4 points] If  $\hat{B}$  is Hermitian, show that  $\langle \hat{B}^2 \rangle \geq 0$ .

(c) [4 points] A Hermitian operator has a discrete set of eigenvalues  $a_n \neq 0$  and eigenfunctions  $\psi_n$ . What are the eigenvalues and eigenfunctions of the operator  $\hat{A}^{-1}$  where  $\hat{A}^{-1}\hat{A} = \hat{I}$ ?

**Problem 2 continued:**

(d) [4 points] Is  $\hat{L}_-$  an observable operator? Justify your answer.

(e) [4 points] Is  $\hat{L}_+\hat{L}_-$  an observable operator? Justify your answer.

(f) [4 points] Calculate  $\hat{L}_+\hat{L}_-|\ell m_\ell\rangle$ .

(g) [4 points] Based solely on the result of the previous part does  $\hat{L}_+\hat{L}_-$  commute with  $\hat{L}^2$ ?  
What about with  $\hat{L}_z$ ? Explain your answer.

**Problem 3.** (25 points)

A particle of mass  $m$  is placed in a symmetric infinite square well potential given by

$$V(x) = \begin{cases} \infty, & x \leq -a/2 \\ 0, & -a/2 < x < a/2. \\ \infty, & x \geq a/2 \end{cases}$$

[*Note:* There is a question and an extra credit question on the back of this page.]

(a) [5 points] Sketch the ground state wave function for this potential. Write the (normalized) wave function for the ground state. Is this wave function an even or odd function?

(b) [17 points] Solve for the **unnormalized** ground state wave function for the potential  $\tilde{V}(x) = V(x) + \alpha\delta(x)$  where  $V(x)$  is the potential from the previous part and  $\alpha$  is a positive, real constant. Assume that the ground state wave function has the same symmetry as you found in the previous part. [*Note:* Do not solve for the energy of the ground state.]

**Problem 3 continued:**

- (c) [3 points] Verify that in the limit  $\alpha \rightarrow 0$  your answer in part (b) is consistent with your answer in part (a). [*Note:* Since we haven't normalized the state nor calculated the energy we can't show the states are identical, though it should be plausible.]
- (d) [10 points] **Extra Credit:** Normalize the wave function you found in part (b) and verify that the normalization is also consistent with your wave function from part (a). [*Note:* Do **not** spend time on this problem until you have worked on all the other problems.]

**Problem 4.** (25 points)

The spectrum of the hydrogen atom includes many small but important corrections. One such correction is the fine structure of the hydrogen spectrum that comes from the spin orbit coupling of the electron. In the rest frame of the electron the proton appears to be moving, a moving charge is a current and a current produces a magnetic field, thus the electron is a spin in a magnetic field as we have discussed before. The Hamiltonian associated with this is

$$\hat{H}_{\text{so}} = -\boldsymbol{\mu} \cdot \mathbf{B} = \frac{\alpha \hbar}{2m_e^2 c r^3} \hat{\mathbf{L}} \cdot \hat{\mathbf{S}},$$

where  $\alpha$  is the fine structure constant. Suppose the electron is in the state  $\psi = \psi_{n\ell m} \chi$  where  $\psi_{n\ell m}$  is the usual hydrogen wave function and  $\chi$  is the spin state of the electron. **[There are questions on the back of this page.]**

(a) [3 points] Show that we can write

$$\hat{\mathbf{L}} \cdot \hat{\mathbf{S}} = \frac{1}{2} (\hat{J}^2 - \hat{L}^2 - \hat{S}^2),$$

where  $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$ .

(b) [4 points] For the state given in this problem  $j$ ,  $\ell$ , and  $s$  are all good quantum numbers. Calculate  $(\hat{\mathbf{L}} \cdot \hat{\mathbf{S}})\psi$  in terms of  $j$ ,  $\ell$ , and constants, as appropriate. [Note: We know  $s$  for the electron so use that value.]

(c) [3 points] For a given  $\ell$  what are the possible values for  $j$ ?

**Problem 4 continued:**

(d) [10 points] Show that

$$E_{\text{so}} \equiv \langle \hat{H}_{\text{so}} \rangle = \frac{(E_n)^2}{m_e c^2} \frac{2n [j(j+1) - \ell(\ell+1) - 3/4]}{\ell(2\ell+1)(\ell+1)},$$

where  $E_n$  is the usual hydrogen energy. Note that  $\langle 1/r^3 \rangle = 2/n^3 \ell(2\ell+1)(\ell+1)a_0^{-3}$  for the state given in this problem.

(e) [5 points] Calculate the fractional change in the energy for the state(s) from part (c),  $E_{\text{so}}/E_n$ , for an electron with  $n = 2$  and  $\ell = 1$ . [Note: It helps to know that  $m_e c^2 = 511 \text{ keV}$ .]

**Problem 5.** (25 points)

Suppose that the Hamiltonian of a system is given by

$$\hat{H} = \left( \hat{b}^\dagger \hat{b} + \frac{1}{2} \right) \hbar\omega = \left( \hat{N} + \frac{1}{2} \right) \hbar\omega$$

where  $\hat{N} \equiv \hat{b}^\dagger \hat{b}$  is the number operator,  $\hat{N}|n\rangle = n|n\rangle$  similar to the harmonic oscillator. Here the operators satisfy *anti*-commutation relation

$$\{ \hat{b}, \hat{b}^\dagger \} \equiv \hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b} = 1.$$

[*Note:* This problem explores the creation and annihilation operators for fermions and is an essential piece of “second quantization”. The usual harmonic oscillator operators describe bosons. **There are questions on the back of this page.**]

(a) [7 points] Show that we can write  $\hat{b}|n\rangle = c_n|1-n\rangle$  where  $c_n$  is a constant. [*Hint:* Evaluate  $\hat{H}(\hat{b}|n\rangle)$ .]

(b) [4 points] Show that we can write  $\hat{b}^\dagger|n\rangle = d_n|1-n\rangle$  where  $d_n$  is a constant.

**Problem 5 continued:**

(c) [7 points] Determine the constants  $c_n$  and  $d_n$ . [*Hint:* Calculate  $\langle n|\hat{N}|n\rangle$  in different ways.]

(d) [7 points] Since  $\langle \hat{H} \rangle \geq 0$  what values can  $n$  have? How many states are there?

**Problem 6.** (24 points)

Two identical, non-interacting particles of mass  $m$  are placed at positions  $x_a$  and  $x_b$  in a one dimensional simple harmonic oscillator potential with angular frequency  $\omega$ . **For the cases below write the ground state wave function and determine the ground state energy for the system.** Specify the energy in terms of  $\hbar\omega$ . [*Note:* Specifying the spatial wave function in each case is sufficient. You do not need to write out the form for the spin wave function, though specifying its important properties is necessary! **There are questions on the back of this page.**]

(a) [6 points] Suppose the two particles are spin-1/2 fermions and are in the singlet spin state.

(b) [6 points] Instead suppose the two particles are spin-1/2 fermions and are in the triplet spin state.

**Problem 6 continued:**

(c) [6 points] Suppose the two particles are spin-1 bosons and are in the  $|21\rangle$  spin state.

(d) [6 points] Instead suppose the two particles are spin-1 bosons and are in the  $|10\rangle$  spin state.