Solutions

Problem 1. (28 points)

Clearly state if each of the following could describe a wave function for $-\infty < x < \infty$. Briefly justify your answer. Here A is a constant. [*Note*: **There are questions on the back of this page.**]

(a) [4 points] $\psi(x) = Ae^{ix}$.

No ψ is not normalizable

(b) [4 points] $\psi(x) = Ae^{-x^2}$.

Yes ψ is normalizable and continuous.

(c) [4 points] $\psi(x) = \begin{cases} 0, & x < 0 \\ A\sin(x), & 0 \le x \le \pi \\ 0, & \pi < x \end{cases}$.

Yes ψ is normalizable and continuous.

$\frac{d\psi}{dx}$ is not continuous at $x=0$ and $x=\pi$

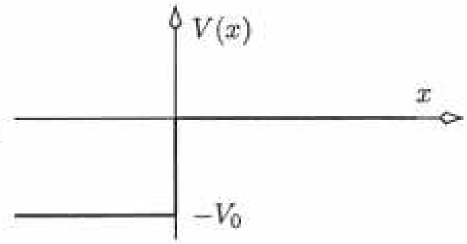
But this is ok if $|V| \to \infty$ there (think infinite square well)

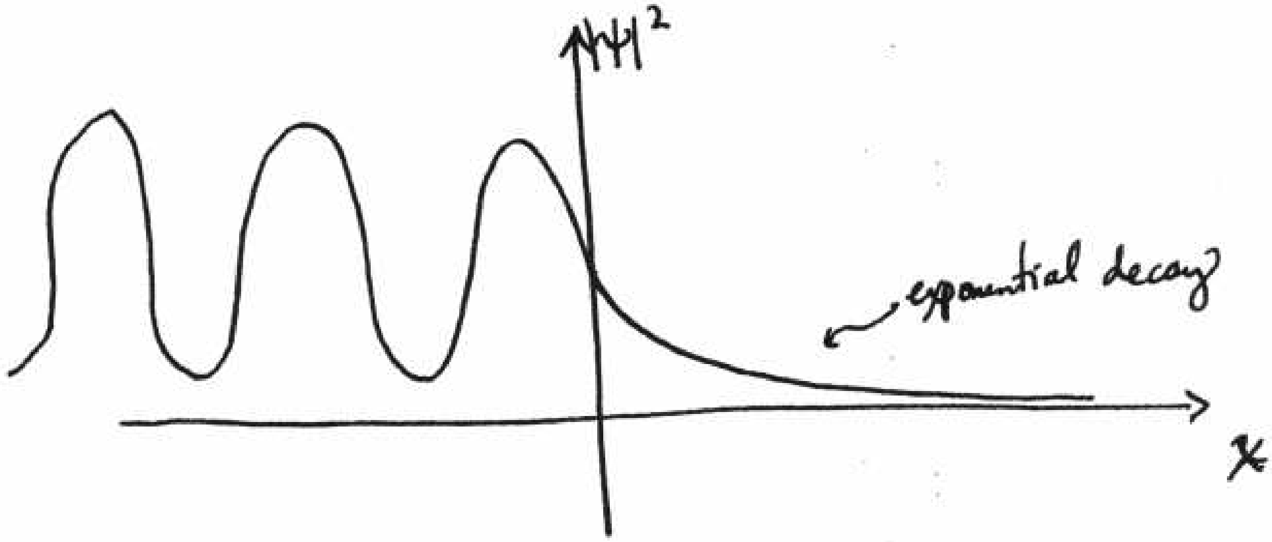
(d) [4 points] $\psi(x) = \begin{cases} 0, & x < 0 \\ A\sin(x), & 0 \le x \le 3\pi/2 \\ 0, & 3\pi/2 < x \end{cases}$.

No ψ is not continuous at $x = 3\pi/2$.

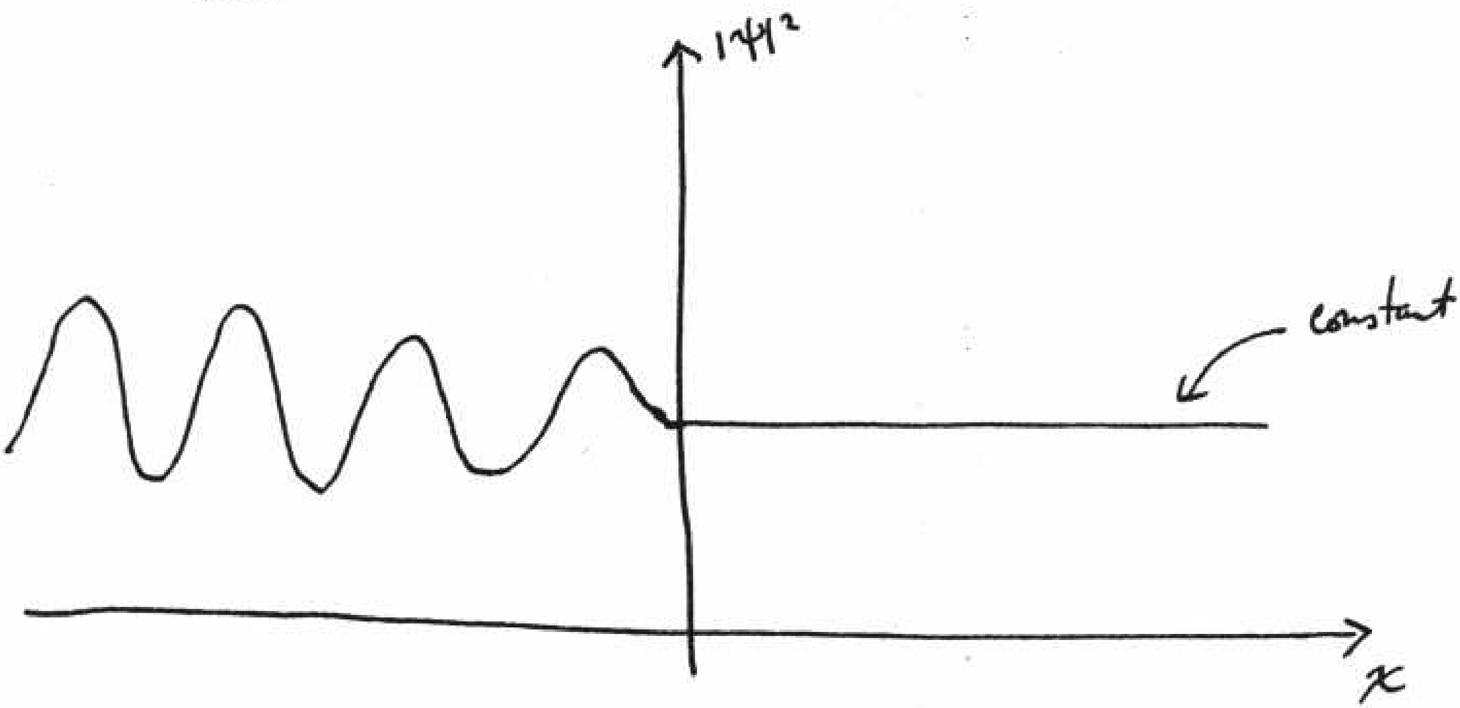
Problem 1 continued:

(e) [6 points] Consider the potential defined by $V(x) = -V_0$ for $x < 0$ and $V(x) = 0$ for $x > 0$ as shown at the right. A particle of energy $-V_0 < E < 0$ is sent in from the left. Sketch the probability density, $|\psi(x)|^2$, for the particle in the regions $x < 0$ and $x > 0$.





(f) [6 points] For the potential in the previous part a particle of energy $E > 0$ is now sent in from the left. Sketch the probability density, $|\psi(x)|^2$, for the particle in the regions $x < 0$ and $x > 0$.



Problem 2. (30 points) Solutions

(a) [4 points] Show that the functions $f(x) = e^x$ and $g(x) = e^{-x}$ are eigenfunctions of the operator d^2/dx^2 with the same eigenvalue. What is the eigenvalue?

$$\frac{d^2f}{dx^2} = \frac{d^2}{dx^2} e^x = \underline{e^x}, \quad \frac{d^2g}{dx^2} = \frac{d^2}{dx^2} e^{-x} = \underline{e^{-x}}$$

So both are eigenfunctions with eigenvalue $\underline{\lambda = +1}$

(b) [4 points] Show that the eigenfunctions in the previous part are not orthogonal on the interval $x \in [-1, 1]$.

$$\underline{\langle f|g\rangle} = \int_{-1}^{1} f^* g\, dx = \int_{-1}^{1} dx = 2 \underline{\neq 0}$$

Thus not orthogonal

(c) [6 points] Construct two linear combinations from the functions $f(x)$ and $g(x)$ that are orthogonal on the interval $x \in [-1, 1]$.

Consider the functions $\boxed{h_\pm = f \pm g = e^x \pm e^{-x}}$

$$\underline{\langle h_+|h_-\rangle} = \int_{-1}^{1} (e^x + e^{-x})(e^x - e^{-x})\, dx$$

$$= \int_{-1}^{1} e^{2x} + 1 - 1 - e^{-2x}\, dx = \int_{-1}^{1} e^{2x} - e^{-2x}\, dx$$

$$= \frac{1}{2}\left(e^{2x} + e^{-2x}\right)\Big|_{-1}^{1} = \frac{1}{2}\left(e^2 + e^{-2} - e^{-2} - e^2\right) = \underline{0}$$

Thus $h_\pm$ are orthogonal.

Problem 2 continued:
A particle with non-zero spin is placed at rest in a region with a constant, uniform magnetic field, $\mathbf{B} = B_0\hat{z}$.

$$\frac{d\langle \hat{Q}\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \left\langle\frac{\partial \hat{Q}}{\partial t}\right\rangle$$

(d) [4 points] Calculate $\frac{d\langle \hat{S}_z\rangle}{dt}$.

$$\hat{H} = -\vec{\mu}\cdot\vec{B} = -\gamma B_0 \hat{S}_z .$$

So $[\hat{H},\hat{S}_z] = 0 \Rightarrow \boxed{\frac{d\langle \hat{S}_z\rangle}{dt} = 0}$

(e) [4 points] Calculate $\frac{d\langle \hat{S}_x\rangle}{dt}$.

$$[\hat{H},\hat{S}_x] = -\gamma B_0[\hat{S}_z,\hat{S}_x] = -i\hbar\gamma B_0 \hat{S}_y$$

$$\Rightarrow \boxed{\frac{d\langle \hat{S}_x\rangle}{dt} = \gamma B_0\langle\hat{S}_y\rangle}$$

(f) [4 points] For the case of an electron verify your result for $\frac{d\langle \hat{S}_z\rangle}{dt}$ from part (a). [*Hint*: We calculated the required expectation values when we studied Larmor precession. You do not need to recalculate them. Feel free to use those results.]

We saw that $\langle\hat{S}_z\rangle = \frac{\hbar}{2}\cos\alpha$ [4.166]

Thus $\frac{d\langle\hat{S}_z\rangle}{dt} = 0$ ✓

(g) [4 points] For the case of an electron verify your result for $\frac{d\langle \hat{S}_x\rangle}{dt}$ from part (b).

We saw that $\langle\hat{S}_x\rangle = \frac{\hbar}{2}\sin\alpha\cos(\gamma B_0 t)$ [4.164]

$\langle\hat{S}_y\rangle = -\frac{\hbar}{2}\sin\alpha\sin(\gamma B_0 t)$ [4.165].

Thus $\frac{d\langle\hat{S}_x\rangle}{dt} = \frac{\hbar}{2}\sin\alpha(-\gamma B_0)\sin(\gamma B_0 t) = -\gamma B_0\frac{\hbar}{2}\sin\alpha\sin(\gamma B_0 t)$

$= \gamma B_0\langle\hat{S}_y\rangle$ ✓

Solutions

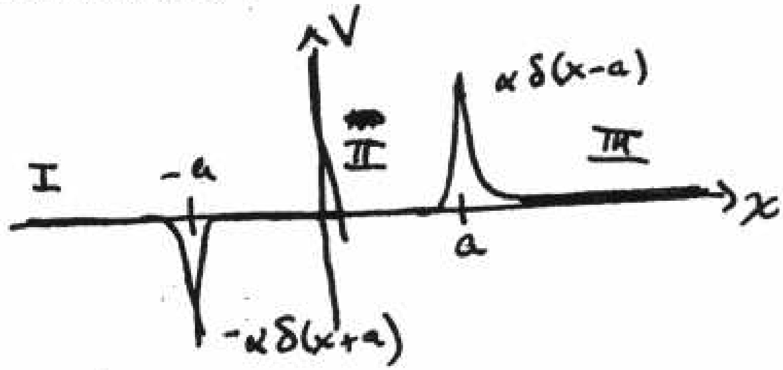
Problem 3. (20 points)
Consider the potential given by $V(x) = -\alpha\delta(x+a) + \alpha\delta(x-a)$ where α and a are positive constants. [*Note*: **There is an extra credit question on the back of this page.**]

(a) [5 points] Can this potential have bound states?

<u>Yes</u> $V(\pm\infty)=0$ and we can have $E=0$.

(b) [15 points] A particle with mass m and energy $E > 0$ is sent in from the left. Write down the equations you would need to solve to find the transmission coefficient for this particle. Do **not** solve for the transmission coefficient but make sure the equations you write down are in a form that could directly be used to solve for it.

We have 3 regions defined by the potential



<u>BC.</u> ψ is continuous between regions. ψ' isn't due to the δ-fcns, $\Delta\psi' = \frac{2m}{\hbar^2}V\psi$ at boundary.

$$\left.\begin{aligned}\psi_{I} &= Ae^{ikx} + Be^{-ikx}\\ \psi_{II} &= Ce^{ikx} + De^{-ikx}\end{aligned}\right\}\ \text{where } k \equiv \sqrt{2mE}.$$

$$\psi_{III} = Fe^{+ikx}$$

Thus

$$\psi_{I}(-a) = \psi_{II}(-a)$$

$$\psi_{II}(a) = \psi_{III}(a)$$

$$\psi'_{II}(-a) - \psi'_{I}(-a) = -\frac{2m}{\hbar^2}\alpha\psi_{I}(-a)$$

$$\psi'_{III}(a) - \psi'_{II}(a) = \frac{2m}{\hbar^2}\alpha\psi_{III}(a)$$

Thus

2 ① $Ae^{-ika} + Be^{ika} = Ce^{-ika} + De^{ika}$

3 ② $Ce^{ika} + De^{-ika} = Fe^{+ika}$

4 ③ $Ce^{-ika} - De^{ika} - Ae^{-ika} + Be^{ika} = -\frac{2m\alpha}{ik\hbar^2}\left(Ae^{-ika} + Be^{+ika}\right)$

4 ④ $Fe^{ika} - Ce^{ika} + De^{-ika} = \frac{2m\alpha}{ik\hbar^2}Fe^{ika}$

Problem 3 continued:

(c) [10 points] **Extra Credit:** Calculate the transmission coefficient for this particle. [*Hint:* The algebra isn't horrible if you combine the equations from the previous part in the right order. Do **not** spend time on this problem until you have worked on all the other problems.]

Briefly we find the following , $\beta \equiv \frac{2m\alpha}{\hbar^2 k}$.

$④ + ② \Rightarrow \quad D = -\frac{i\beta}{2} F e^{2ika}$

$④ - ② \Rightarrow \quad C = \frac{2+i\beta}{2} F$

$① + ③ \Rightarrow \quad B = \frac{i\beta}{2-i\beta}\left(Ae^{-2ika} - Fe^{2ika}\right)$

Plugging these into ① gives

$$A = \frac{1}{4}\left(4+\beta^2 - \beta^2 e^{4ika}\right)F$$

Thus $\underline{\frac{1}{T} = \left|\frac{A}{F}\right|^2 = \frac{1}{8}\left[2 + (2+\beta^2)^2 - \beta^4 (4+\beta^2)\cos(4ka)\right]}$

Solutions

Problem 4. (25 points)

Neutrinos are fundamental particles that are nearly massless. There are three types, also known as flavors, of neutrinos, one associated with each of the fundamental leptons (electron, muon, and tau). If the flavor states of the neutrino are not eigenstates of the Hamiltonian then we can have oscillations between the types of neutrinos. To explore this we will consider just two neutrino flavors and label the flavor states as $|\nu_\mu\rangle$ and $|\nu_\tau\rangle$ (if the third flavor state is much lighter than these two then it makes sense to treat these two together). [*Note*: **There are questions on the back of this page.**]

(a) [4 points] Label the energy eigenstates as $|\nu_1\rangle$ and $|\nu_2\rangle$ then $\hat{H}|\nu_1\rangle = E_1|\nu_1\rangle$ and $\hat{H}|\nu_2\rangle = E_2|\nu_2\rangle$. Write the Hamiltonian in matrix form in the $\{|\nu_1\rangle, |\nu_2\rangle\}$ basis.

$$\underline{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

(b) [6 points] The flavor eigenstates are typically written in terms of the energy eigenstates as $|\nu_\mu\rangle = \cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle$ and $|\nu_\tau\rangle = -\sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle$ where θ is called the mixing angle. We can transform from one basis to the other using a matrix U where

$$\begin{pmatrix} |\nu_\mu\rangle \\ |\nu_\tau\rangle \end{pmatrix} = \mathsf{U}\begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}.$$

Write down the matrix U and show that it is unitary.

$$\underline{U} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

To be unitary $\underline{U}^\dagger = \underline{U}^{-1} \Rightarrow \underline{U}\,\underline{U}^\dagger = \underline{U}^\dagger\underline{U} = \underline{I}$.

$$\underline{U}\,\underline{U}^\dagger = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

$$\underline{U}^\dagger\underline{U} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

Problem 4 continued:

(c) [8 points] In a particle decay the flavor eigenstates are formed. Suppose we start in the state $|\mathcal{S}(0)\rangle = |\nu_\mu\rangle$. Write down $|\mathcal{S}(t)\rangle$ in the $\{|\nu_\mu\rangle, |\nu_\tau\rangle\}$ basis.

$$|S(0)\rangle = |\nu_\mu\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle$$

Thus $$|S(t)\rangle = \cos\theta\, e^{-iE_1t/\hbar}|\nu_1\rangle + \sin\theta\, e^{-iE_2t/\hbar}|\nu_2\rangle$$

$$= \left(\cos^2\theta\,|\nu_\mu\rangle - \cos\theta\sin\theta\,|\nu_\tau\rangle\right)e^{-iE_1t/\hbar} + \left(\sin^2\theta\,|\nu_\mu\rangle + \sin\theta\cos\theta\,|\nu_\tau\rangle\right)e^{-iE_2 t/\hbar}$$

$$\boxed{|S(t)\rangle = \left(\cos^2\theta\, e^{-iE_1t/\hbar} + \sin^2\theta\, e^{-iE_2t/\hbar}\right)|\nu_\mu\rangle + \sin\theta\cos\theta\left(e^{-iE_2t/\hbar} - e^{-iE_1t/\hbar}\right)|\nu_\tau\rangle}$$

(d) [7 points] For the initial state in the previous part show that the probability of detecting the neutrino in the $|\nu_\tau\rangle$ state as a function of time, $P_\tau(t)$, is given by

$$P_\tau(t) = \sin^2(2\theta)\sin^2(\omega t)$$

where $E_2 - E_1 \equiv 2\hbar\omega$.

$$P_\tau(t) = |\langle\nu_\tau|S(t)\rangle|^2 = \left|\sin\theta\cos\theta\, e^{-iE_2t/\hbar}\left(1 - e^{i(E_2-E_1)t/\hbar}\right)\right|^2$$

$$= \left|\tfrac{1}{2}\sin 2\theta\right|^2\left(1 + 1 - e^{2i\omega t} - e^{-2i\omega t}\right) = \tfrac{1}{4}\sin^2 2\theta\,(2 - 2\cos 2\omega t)$$

$$= \tfrac{1}{2}\sin^2 2\theta\,(1 - \cos 2\omega t)$$

But $\cos 2\omega t = \cos^2\omega t - \sin^2\omega t = 1 - 2\sin^2\omega t$.

Thus $1 - \cos 2\omega t = 2\sin^2\omega t$.

Thus $\boxed{P_\tau(t) = \sin^2 2\theta\,\sin^2\omega t}$

Solutions

Problem 5. (34 points)

Frequently in a two particle system the interaction (potential energy) will only depend on the difference in their positions, $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$. In this case the Schrödinger equation separates if we write it in terms of $\mathbf{r}$ and the center of mass coordinate, $\mathbf{R} \equiv (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/(m_1+m_2)$. Using this transformation it can be shown that $\nabla_1 = (\mu/m_2)\nabla_R + \nabla_r$, and $\nabla_2 = (\mu/m_1)\nabla_R - \nabla_r$ where $\mu \equiv m_1 m_2/(m_1+m_2)$ is the reduced mass. [*Hint:* See problem 5.1 in the text book. *Note:* **There are questions on the back of this page.**]

(a) [8 points] For this system show that the (time-independent) Schrödinger equation becomes

$$-\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2\psi - \frac{\hbar^2}{2\mu}\nabla_r^2\psi + V(\mathbf{r})\psi = E\psi. \leftarrow \hat{H}\psi = E\psi.$$

$$\hat{H} = -\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + V(\vec{r}).$$

$$= -\frac{\hbar^2}{2}\left[\frac{1}{m_1}\left(\frac{\mu}{m_2}\vec{\nabla}_R + \vec{\nabla}_r\right)^2 + \frac{1}{m_2}\left(\frac{\mu}{m_1}\vec{\nabla}_R - \vec{\nabla}_r\right)^2\right] + V(\vec{r})$$

$$= -\frac{\hbar^2}{2}\left[\frac{\mu^2}{m_1m_2}\underbrace{\left(\frac{1}{m_1}+\frac{1}{m_2}\right)}_{\mu}\nabla_R^2 + \underbrace{\left(\frac{1}{m_1}+\frac{1}{m_2}\right)}_{\mu}\nabla_r^2 + \cancel{\frac{2\mu}{m_1m_2}\left(\vec{\nabla}_R\cdot\vec{\nabla}_r - \vec{\nabla}_R\cdot\vec{\nabla}_r\right)}\right] + V(\vec{r})$$

$$= -\frac{\hbar^2}{2}\left[\frac{1}{m_1+m_2}\nabla_R^2 + \mu\nabla_r^2\right] + V(\vec{r}).$$ Thus we get the desired result.

(b) [6 points] Use separation of variables, $\psi(\mathbf{R},\mathbf{r}) = \psi_R(\mathbf{R})\psi_r(\mathbf{r})$, to write down the equations that ψ_R and ψ_r must satisfy.

Plugging in we have $-\frac{\hbar^2}{2(m_1+m_2)}\frac{\nabla_R^2\psi_R}{\psi_R} - \frac{\hbar^2}{2\mu}\frac{\nabla_r^2\psi_r}{\psi_r} + V(\vec{r}) = E.$

Thus

$$\boxed{\begin{aligned} -\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2\psi_R &= E_R\psi_R \\ -\frac{\hbar^2}{2\mu}\nabla_r^2\psi_r + V(\vec{r})\psi_r &= E_r\psi_r \end{aligned}}$$

where $E = E_R + E_r$.

(c) [6 points] For the case of the hydrogen atom where $V(\mathbf{r}) = -e^2/4\pi\epsilon_0 r$ write down an expression for the eigenstates $\psi_r(\mathbf{r})$. [*Note:* You recognize this system so write down or describe the solution in some way that I can recognize that you know the solution.]

This is exactly the case of the hydrogen atom with $m_e \rightarrow \mu$. Thus the eigenstates are $\boxed{\psi_r = \psi_{nlm}}$ as we found before

See eqn [4.89]

Problem 5 continued:

(d) [6 points] Discuss the solutions $\psi_R(\mathbf{R})$ which describe the motion of center of mass of the hydrogen atom. What difficulties do we encounter? [*Note*: This is identical to a case we encountered in one dimension. You can either discuss the solutions based on physical arguments or try to solve the differential equation. If you attempt to solve the differential equation then only consider the $\ell = 0$ case.]

This is Schrödinger's equation for a free particle in 3-d. Thus we expect the eigenstates to be of the form $\underline{\psi_R \sim e^{\pm i\vec{k}\cdot\vec{R}}}$. [Actually we expect them to be like $e^{\pm i\vec{k}\cdot\vec{R}}/R$.] But just as in 1-d this is not normalizable. Thus we really need to treat the CM of the system as a wave packet just as we would any free particle.

(e) [4 points] Using these results find the percent error in the binding energy of the hydrogen atom introduced by our use of m_e instead of μ. [*Note*: $m_p = 1836 m_e$.]

$E_1 = -m_e \beta$ where β are constants.

Thus $\dfrac{|E_1^{approx} - E_1^{\mu}|}{E_1^{\mu}} = \dfrac{|\mu - m_e|}{\mu} = \left|1 - \dfrac{m_e}{\mu}\right| = \left|1 - \dfrac{\cancel{m_e}(m_p + m_e)}{m_p \cancel{m_e}}\right| = \left|1 - 1 - \dfrac{m_e}{m_p}\right|$

$\Rightarrow$ $\boxed{\text{error is } \dfrac{m_e}{m_p} = 5.45\times 10^{-4} = 0.0545\,\%}$

(f) [4 points] Using these results recalculate the ground state energy of the muonic atom. Recall that $m_\mu = 207 m_e$. [*Note*: This result is much more accurate than our estimate on exam III.]

$\underline{E_{\mu\,atom}} = \dfrac{\mu}{m_e} E_1 = \dfrac{m_p m_\mu}{m_e(m_p + m_\mu)}(-13.6\,eV) = -2529\,eV$

$= \underline{-2.529\,keV}$