Problem 1. (20 points)

(a) You place a particle with energy $E = \hbar^2\beta_{21}^2/2ma^2$ in an infinite spherical well. Write down the most general wavefunction, $\psi(\mathbf{r})$, for this particle. [*Hint*: Write it in terms of $\psi_{nlm}(\mathbf{r})$.]

In this state $n=2$, $l=1$, $m=-1, 0,$ or 1. Thus

$$\psi(\vec{r}) = a\,\psi_{21-1} + b\,\psi_{210} + c\,\psi_{211}$$

where $|a|^2+|b|^2+|c|^2=1$.

(b) A muonic hydrogen atom is made by replacing the electron in the hydrogen atom with a muon. A muon is identical to an electron (same charge, spin, *etc.*) but is much more massive, $m_\mu = 207m_e$. Calculate the ground state energy in electron volts of the muonic hydrogen atom. Calculate the Bohr radius in meters of the muonic hydrogen atom. [*Note*: Some of the assumptions we made in calculating the hydrogen atom break down when we include a muon instead of the electron. Ignore these details for this calculation.]

$$E_1 = -\frac{m_\mu}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 = \frac{m_\mu}{m_e}\left[-\frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right] = -13.6\,\text{eV}\,\frac{m_\mu}{m_e}$$

So $\boxed{E_1 = -2815\,\text{eV} = -2.82\,\text{keV}}$

$$\underline{a_0} = \frac{4\pi\epsilon_0\hbar^2}{m_\mu e^2} = \frac{m_e}{m_\mu}\left[\frac{4\pi\epsilon_0\hbar^2}{m_e e^2}\right] = \frac{m_e}{m_\mu}\left(0.529\times10^{-10}\,\text{m}\right) = \underline{2.55\times10^{-13}\,\text{m}}$$

(c) Calculate the uncertainty relation $\sigma_z\sigma_{L_z}$.

$$\sigma_z\sigma_{L_z} \geq \left|\frac{1}{2i}\langle[\hat{z},\hat{L}_z]\rangle\right|. \quad \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x.$$

Thus $[\hat{z},\hat{L}_z]=0 \Rightarrow \boxed{\sigma_z\sigma_{L_z}\geq 0}$

(d) Calculate the uncertainty relation $\sigma_x\sigma_{L_z}$.

$$[\hat{x},\hat{L}_z] = [\hat{x}, \hat{x}\hat{p}_y - \hat{y}\hat{p}_x] = -\hat{y}[\hat{x},\hat{p}_x] = -i\hbar\hat{y}.$$

So $\boxed{\sigma_x\sigma_{L_z} \geq \frac{\hbar}{2}|\langle\hat{y}\rangle|}$

Problem 2. (20 points)
A hydrogen atom is in the state $n = 3$, $\ell = 2$, $m = 1$. [*Note*: **There is a question on the back of this page.**]

(a) [6 points] Write down the wave function for this hydrogen atom. Express your answer in terms of r, θ, ϕ, and constants, as appropriate.

$$\psi_{321}(\vec{r}) = R_{32}(r)\, Y_2^1(\theta, \varphi)$$

$$= \frac{4}{81\sqrt{30}}\, a_0^{-3/2} \left(\frac{r}{a_0}\right)^2 e^{-r/3a_0} \left[-\sqrt{\frac{15}{8\pi}}\, \sin\theta\cos\theta\, e^{i\varphi}\right]$$

so $$\boxed{\psi_{321}(\vec{r}) = -\frac{a_0^{-3/2}}{81\sqrt{\pi}} \left(\frac{r}{a_0}\right)^2 e^{-r/3a_0} \sin\theta\cos\theta\, e^{i\varphi}}$$

(b) [8 points] Find the expectation value of r for this hydrogen atom.

$$\langle r \rangle = \int r\, |\psi_{321}|^2 d^3r = \int_0^\infty r\, |R_{32}|^2 r^2 dr$$

$$= \frac{2^4}{3^8\, 30}\, a_0^{-3} \frac{1}{a_0^4} \int_0^\infty r^3 r^4 e^{-2r/3a_0} dr$$

$$= \frac{2^4}{3^8\, 30} \frac{1}{a_0^7}\, 7! \left(\frac{3a_0}{2}\right)^8 = \frac{\cancel{2^4} \cdot 7 \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot 3 \cdot \cancel{2}\ a_0}{\cancel{3^8} \cdot \cancel{2 \cdot 3} \cdot \cancel{5}\ 2}$$

$$\boxed{\langle r \rangle = \frac{21}{2} a_0}$$

Problem 2 continued:

(c) [6 points] If we could measure the observable $\hat{L}_x^2 + \hat{L}_y^2$ for this hydrogem atom what value (or values) could we find and with what probability of each?

$$\hat{L}^2 \psi_{321} = 2\cdot 3\hbar^2 \psi_{321} = 6\hbar^2 \psi_{321}$$

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

$$\Rightarrow \left(\hat{L}_x^2 + \hat{L}_y^2\right)\psi_{321} = \left(\hat{L}^2 - \hat{L}_z^2\right)\psi_{321} = \left(6\hbar^2 - \hbar^2\right)\psi_{321}$$

$$= 5\hbar^2 \psi_{321}.$$

We measure $5\hbar^2$ with 100% prob

Problem 3. (20 points)
A hydrogen atom contains two spin 1/2 particles. As discussed in class the total spin of the hydrogen can be either 0 (singlet state) or 1 (triplet state) when the atom is in the ground state. Now consider the case when the hydrogen atom is in the $\ell = 2$ state. Here we will look at the total angular momentum $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$. Consider the case where the spins of the hydrogen atom are found to be in the triplet state and a measurement of $\hat{J}_z$ tells us that $m_j = 0$. [*Note*: **There are questions on the back of this page.**]

(a) [4 points] What are the possible values for the total angular momentum quantum number j?

$s=1, \ \ell=2$ so $\boxed{j = 1, 2, 3}$

(b) [8 points] For each of the possible values for j write the state $|jm_j\rangle$ in terms of the individual states $|\ell m_\ell\rangle|sm_s\rangle$.

2×1

$|10\rangle = \sqrt{\frac{3}{10}}\,|2\,1\rangle|1\,{-1}\rangle - \sqrt{\frac{2}{5}}\,|20\rangle|10\rangle + \sqrt{\frac{3}{10}}\,|2\,{-1}\rangle|11\rangle.$

$|20\rangle = \sqrt{\frac{1}{2}}\,|21\rangle|1\,{-1}\rangle + 0 - \frac{1}{\sqrt{2}}\,|2\,{-1}\rangle|11\rangle$

$|30\rangle = \sqrt{\frac{1}{5}}\,|21\rangle|1\,{-1}\rangle + \sqrt{\frac{3}{5}}\,|20\rangle|10\rangle + \sqrt{\frac{1}{5}}\,|2\,{-1}\rangle|11\rangle.$

Problem 3 continued:

(c) [4 points] If $j = 1$ what is the probability of measuring $m_s = 1$? What is the probability of measuring $m_\ell = 0$?

$m_s = 1$ prob $3/10$

$m_\ell = 0$ prob $2/5$

(d) [4 points] If we do not measure j then it can have any value from part (a). In this case what is the total probability that a measurement will give $m_\ell = 0$? [*Note*: Assume each value of j is equally probable.]

$\psi = \frac{1}{\sqrt{3}}\left(|10\rangle + |20\rangle + |30\rangle\right)$

So $m_\ell = 0$ with prob $\frac{1}{3}\left(\frac{2}{5} + \frac{3}{5}\right) = \underline{\frac{1}{3}}$