Solutions

Problem 1. (20 points)
For a particle of mass m moving in one dimensional potential $V(x)$ calculate the following. **There is a question on the back of this page.**

(a) [5 points] Uncertainty relation for $\sigma_x\sigma_H$. $\sigma_A\sigma_B \geq \left|\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right|$

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}).$$

So $$[\hat{x},\hat{H}]f(x) = \frac{1}{2m}[\hat{x},\hat{p}^2]f(x) = \frac{1}{2m}\frac{\hbar}{i}\left[x\frac{d^2}{dx^2}f - \frac{d^2}{dx^2}(xf)\right]$$

$$= \frac{1}{2m}\left(\frac{\hbar}{i}\right)^2\left[xf'' - \frac{d}{dx}(xf' + f)\right] = \frac{1}{2m}\left(\frac{\hbar}{i}\right)^2\left[xf'' - xf'' - f' - f'\right].$$

$$= -\frac{1}{m}\left(\frac{\hbar}{i}\right)^2\left(\frac{d}{dx}f\right) = -\frac{\hbar\hat{p}}{im}f.$$

So $[\hat{x},\hat{H}] = -\frac{\hbar}{im}\hat{p}. \Rightarrow \boxed{\sigma_x\sigma_H \geq \frac{\hbar}{2m}|\langle\hat{p}\rangle|}$

OR use $\frac{d\langle\hat{x}\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H},\hat{x}]\rangle + \left\langle\frac{\partial\hat{x}}{\partial t}\right\rangle \Rightarrow \langle[\hat{H},\hat{x}]\rangle = \frac{\hbar}{i}\frac{d\langle\hat{x}\rangle}{dt} = \frac{\hbar}{i}\langle\hat{p}\rangle.$

$\Rightarrow \sigma_x\sigma_H \geq \frac{\hbar}{2}|\langle\hat{p}\rangle|.$

(b) [5 points] Uncertainty relation for $\sigma_p\sigma_H$.

$$[\hat{p},\hat{H}]f = [\hat{p},V(\hat{x})]f = \frac{\hbar}{i}\left[\frac{d}{dx}(Vf) - V\frac{d}{dx}f\right]$$

$$= \frac{\hbar}{i}\left[\frac{dV}{dx}f + V\frac{df}{dx} - V\frac{df}{dx}\right] = \frac{\hbar}{i}\frac{dV}{dx}f$$

$\Rightarrow \boxed{\sigma_p\sigma_H \geq \frac{\hbar}{2}\left|\left\langle\frac{dV}{dx}\right\rangle\right|}$

OR as above $\langle[\hat{H},\hat{p}]\rangle = \frac{\hbar}{i}\frac{d\langle\hat{p}\rangle}{dt}$
$= \frac{\hbar}{i}\left\langle\frac{\partial V}{\partial x}\right\rangle$ by Ehrenfest's theorem.

(c) [5 points] Uncertainty relation for $\sigma_x\sigma_T$. [*Note*: $\hat{T}$ is the kinetic energy of the particle.]

$$[\hat{x},\hat{T}] = \left[\hat{x},\frac{\hat{p}^2}{2m}\right] = [\hat{x},\hat{H}] = -\frac{\hbar}{i}\frac{p}{m}$$

Thus. $\boxed{\sigma_x\sigma_T \geq \frac{\hbar}{2m}|\langle\hat{p}\rangle|}$

Problem 1 continued:

(d) [5 points] Show that $m\dfrac{d\langle\hat{x}^2\rangle}{dt} = \langle\hat{p}\hat{x}\rangle + \langle\hat{x}\hat{p}\rangle = \langle\{\hat{p},\hat{x}\}\rangle$.

$$\frac{d\langle\hat{x}^2\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H},\hat{x}^2]\rangle + \underbrace{\cancel{\left\langle\frac{\partial\hat{x}^2}{\partial t}\right\rangle}}_{0}$$

$$[\hat{H},\hat{x}^2] = \frac{1}{2m}[\hat{p}^2,\hat{x}^2].$$

So we need $[\hat{p}^2,\hat{x}^2]$. This can be found in a number of ways.

Here we use $[\hat{x},\hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$.

Then

$$\begin{aligned}
[\hat{p}^2,\hat{x}^2] &= \hat{p}\hat{p}\hat{x}\hat{x} - \hat{x}\hat{x}\hat{p}\hat{p} = \hat{p}(\hat{x}\hat{p}-i\hbar)\hat{x} - \hat{x}(\hat{p}\hat{x}+i\hbar)\hat{p} \\
&= \hat{p}\hat{x}\hat{p}\hat{x} - \hat{x}\hat{p}\hat{x}\hat{p} - i\hbar(\hat{p}\hat{x}+\hat{x}\hat{p}) \\
&= \hat{p}\hat{x}(\hat{x}\hat{p}-i\hbar) - (\hat{p}\hat{x}+i\hbar)\hat{x}\hat{p} - i\hbar(\hat{p}\hat{x}+\hat{x}\hat{p}) \\
&= \cancel{\hat{p}\hat{x}\hat{x}\hat{p}} - \cancel{\hat{p}\hat{x}\hat{x}\hat{p}} - 2i\hbar(\hat{p}\hat{x}+\hat{x}\hat{p}) \\
&= -2i\hbar(\hat{p}\hat{x}+\hat{x}\hat{p}).
\end{aligned}$$

Thus $[\hat{H},\hat{x}^2] = -\dfrac{i\hbar}{m}(\hat{p}\hat{x}+\hat{x}\hat{p})$.

So
$$\boxed{m\frac{d\langle\hat{x}^2\rangle}{dt} = \langle\hat{p}\hat{x}+\hat{x}\hat{p}\rangle = \langle\{\hat{p},\hat{x}\}\rangle}$$

Solutions

Problem 2. (40 points)

Consider the Hamiltonian for a 2-level system

$$\hat{H} = h|1\rangle\langle 1| + ig|1\rangle\langle 2| - ig|2\rangle\langle 1| + h|2\rangle\langle 2|$$

written in Dirac notation in terms of the orthonormal basis $\{|1\rangle, |2\rangle\}$. Here h and g are real numbers and $g \neq 0$. **There are questions on the back of this page.**

(a) [5 points] Write the Hamiltonian in matrix form in the $\{|1\rangle, |2\rangle\}$ basis.

$H_{mn} = \langle m|\hat{H}|n\rangle$

$$\boxed{\underline{H} = \begin{pmatrix} h & ig \\ -ig & h \end{pmatrix}}$$

(b) [5 points] Find the eigenvalues for this system.

$$|\underline{H} - E\underline{I}| = 0 = \begin{vmatrix} h-E & ig \\ -ig & h-E \end{vmatrix} = (h-E)^2 - g^2 = 0$$

$$\Rightarrow h - E = \pm g \Rightarrow \boxed{E_\pm = h \pm g}$$

(c) [10 points] Find the eigenvectors for this system. Also write them in the basis $\{|1\rangle, |2\rangle\}$.

$$\underline{H}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = E_\pm \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \Rightarrow \cancel{h a_1} + ig a_2 = (\cancel{h} \pm g) a_1 \Rightarrow \underline{a_2 = \mp i a_1}$$

so $\boxed{|e_\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mp i \end{pmatrix}}$ ← There are other ways to write this. The i could be in upper component, for example.

so $\boxed{|e_\pm\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle \mp i|2\rangle\right)}$

Problem 2 continued:

(d) [5 points] Suppose the system starts in the state $|\mathcal{S}(0)\rangle = (|1\rangle + 2|2\rangle)/\sqrt{5}$. What are the possible energy states that can be measured and with what probability?

Possible energy states are $\boxed{E_\pm = h \pm g}$

$$C_\pm = \langle S(0) | e_\pm \rangle = \frac{1}{\sqrt{2}\sqrt{5}} (1, 2) \begin{pmatrix} 1 \\ \mp i \end{pmatrix} = \frac{1 \mp 2i}{\sqrt{10}}.$$

So $|C_\pm|^2 = \frac{1+4}{10} = \frac{5}{10} = \frac{1}{2}.$

$\Rightarrow$ 50% probability of measuring either ~~at~~ Energy E_+ or E_-

(e) [10 points] Suppose the system starts in the state $|\mathcal{S}(0)\rangle = |2\rangle$. What is the state of the system, $|\mathcal{S}(t)\rangle$, at some time t? Write your answer in the basis $\{|1\rangle, |2\rangle\}$.

$$|S(0)\rangle = |2\rangle = \frac{i}{\sqrt{2}} \left(|e_+\rangle - |e_-\rangle \right).$$

So
$$|S(t)\rangle = \frac{i}{\sqrt{2}} \left(e^{-i(h+g)t/\hbar} |e_+\rangle - e^{-i(h-g)t/\hbar} |e_-\rangle \right)$$
$$= \frac{i}{\sqrt{2}} e^{-iht/\hbar} \left(e^{-igt/\hbar} |e_+\rangle - e^{igt/\hbar} |e_-\rangle \right)$$

In terms of $\{|1\rangle, |2\rangle\}$:

$$|S(t)\rangle = \frac{i}{\sqrt{2}\sqrt{2}} e^{-iht/\hbar} \left[e^{-igt/\hbar} (|1\rangle - i|2\rangle) - e^{igt/\hbar} (|1\rangle + i|2\rangle) \right]$$
$$= \frac{i}{2} e^{-iht/\hbar} \left[-(e^{igt/\hbar} - e^{-igt/\hbar})|1\rangle - i(e^{igt/\hbar} + e^{-igt/\hbar})|2\rangle \right]$$

$$\boxed{|S(t)\rangle = e^{-iht/\hbar} \left[\sin(gt/\hbar)\, |1\rangle + \cos(gt/\hbar)\, |2\rangle \right]}$$