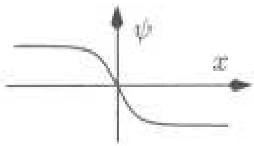
Solutions

Problem 1. (20 points)

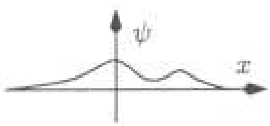
For the following four wavefunctions, $\psi(x)$, shown in the sketches state whether these could describe physical states. Clearly explain your reasoning. [*Note*: The wavefunctions extend to $x \to \pm\infty$.] **There are questions on the back of this page.**



(a)

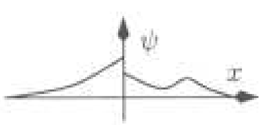
A wave function must be normalizable, continuous, and its first derivative continuous (unless ~~there is a~~ $V \to \pm\infty$ somewhere).

$\boxed{\text{No, Not normalizable}}$



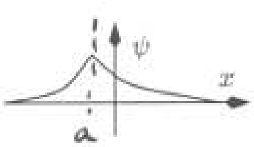
(b)

$\boxed{\text{Yes, satisfies all the conditions}}$



(c)

$\boxed{\text{No, not continuous}}$



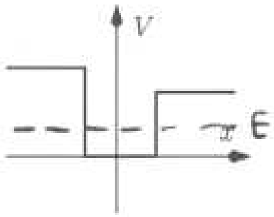
(d)

$\boxed{\text{Yes, if there is a } \delta\text{-function potential at the cusp } (x = a)\text{. Otherwise no.}}$

Problem 1 continued:

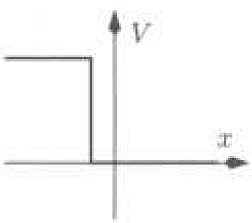
For the following four potentials, $V(x)$, shown in the sketches state whether these could have bound states. Clearly explain your reasoning. [*Note*: The potentials extend to $x \to \pm\infty$. You do **not** need to show that a potential does have a bound state, you just have to explain whether it could have at least one.]

To have bound states we need to have states with $E < V(x \to \pm\infty)$.



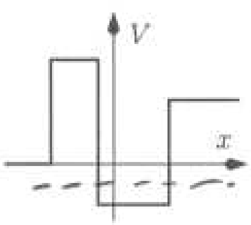
(e)

$\boxed{\text{Yes for E shown.}}$



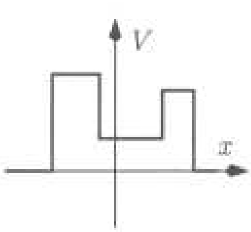
(f)

$\boxed{\text{No, } E \geq 0 \text{ but } V(+\infty) \to 0}$



(g)

$\boxed{\text{Yes for } E < 0}$



(h)

$\boxed{\text{No, } E > 0 \text{ but } V(\pm\infty) \to 0}$

Solutions

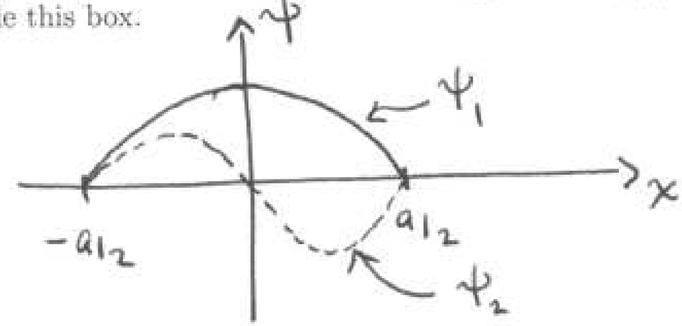
Problem 2. (20 points)

Consider the symmetric infinite square well potential

$$V(x) = \begin{cases} \infty, & x \le -a/2 \\ 0, & -a/2 < x < a/2 \\ \infty, & x \ge a/2 \end{cases} .$$

Answer the following questions **without** doing any calculations. If you feel the need to do a calculation then you will be spending way too much time on this problem!

(a) [4 points] Sketch the wave function of the two lowest energy states $\psi_1(x)$ and $\psi_2(x)$ for a particle of mass m inside this box.



(b) [4 points] What are the energies of these two states, E_1 and E_2? Express your answer in terms of m, a, and constants, as appropriate.

Same as square well from book: $E_n = \dfrac{n^2\hbar^2\pi^2}{2ma^2}$

So $\boxed{E_1 = \dfrac{\pi^2\hbar^2}{2ma^2}, \quad E_2 = \dfrac{2\pi^2\hbar^2}{ma^2}}$

(c) [8 points] Write down expressions for the wavefunctions, $\psi_1(x)$ and $\psi_2(x)$. They should be normalized.

$$\boxed{\begin{aligned} \psi_1(x) &= \sqrt{\frac{2}{a}}\, \overset{\cos}{\cancel{\sin}} \left(\frac{\pi}{a}x\right) \\ \psi_2(x) &= -\sqrt{\frac{2}{a}}\, \sin\left(\frac{2\pi}{a}x\right) \end{aligned}}$$

to be consistent with my sketch in part (a)

(d) [4 points] Write down expressions for the time dependent wavefunctions, $\Psi_1(x,t)$ and $\Psi_2(x,t)$. They should be normalized.

$\Psi_n(x,t) = \psi_n(x)\, e^{-iE_nt/\hbar}$

So
$$\boxed{\begin{aligned} \Psi_1(x,t) &= \sqrt{\frac{2}{a}}\cos\left(\frac{\pi}{a}x\right) e^{-\frac{i\pi^2\hbar}{2ma^2}t} \\ \Psi_2(x,t) &= -\sqrt{\frac{2}{a}}\sin\left(\frac{2\pi}{a}x\right) e^{-i\frac{2\pi^2\hbar}{ma^2}t} \end{aligned}}$$

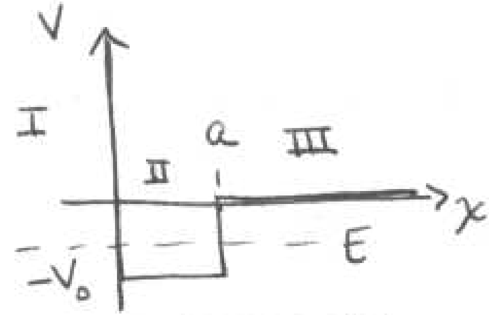
Solutions

Problem 3. (20 points)
Consider the potential

$$V(x) = \begin{cases} \infty, & x < 0 \\ -V_0, & 0 \le x \le a \\ 0, & x > a \end{cases} .$$

There is a question on the back of this page.

(a) [14 points] Solve the time independent Schrödinger equation for $\psi(x)$ when $-V_0 < E < 0$. Clearly state what boundary conditions you are using. Do **not** normalize the wavefunction, leave it in terms of one constant you could find via normalization. You should also find a transcendental equation which determines the allowed bound state energies. Clearly point out this equation.



$\psi_I(x) = 0$ since $V(x<0) = \infty$.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_{II}}{dx^2} - V_0\psi_{II} = E\psi_{II}$$

So $\psi_{II}(x) = A\sin kx + B\cos(kx)$ where $k = \frac{\sqrt{2m(E+V_0)}}{\hbar}$.
(could write as $e^{\pm ikx}$ if desired)

$\psi_{III}(x) = Ce^{-Kx} + De^{+Kx}$ where $K = \frac{\sqrt{-2mE}}{\hbar}$

$D = 0$ to be normalizable.

B.C.

continuity $\begin{cases} \psi_{II}(0) = \psi_I(0) = \boxed{0 = B} \\ \psi_{II}(a) = \psi_{III}(a) \Rightarrow A\sin ka = Ce^{-Ka} \Rightarrow \boxed{C = Ae^{Ka}\sin ka} \end{cases}$

So

$$\boxed{\psi(x) = \begin{cases} 0, & x < 0 \\ A\sin kx, & 0 \le x \le a \\ A\sin ka\, e^{-K(x-a)}, & x > a \end{cases}}$$

$\psi_{II}'(a) = \psi_{III}'(a) \Rightarrow k\cancel{A}\cos ka = -K\cancel{A}\sin ka$

$\Rightarrow \boxed{\cot ka = -\frac{K}{k}}$ ← trancendental equation

Problem 3 continued:

(b) [6 points] You should find that there isn't always a bound state solution. Find the minimum value of V_0 required to have at least one bound state.

To find the minimum V_0 we look at the trancendental equation

$$\cot ka = -\frac{K}{k}.$$

Let $z \equiv ka$ and $z_0 \equiv \frac{a\sqrt{2mV_0}}{\hbar}$.

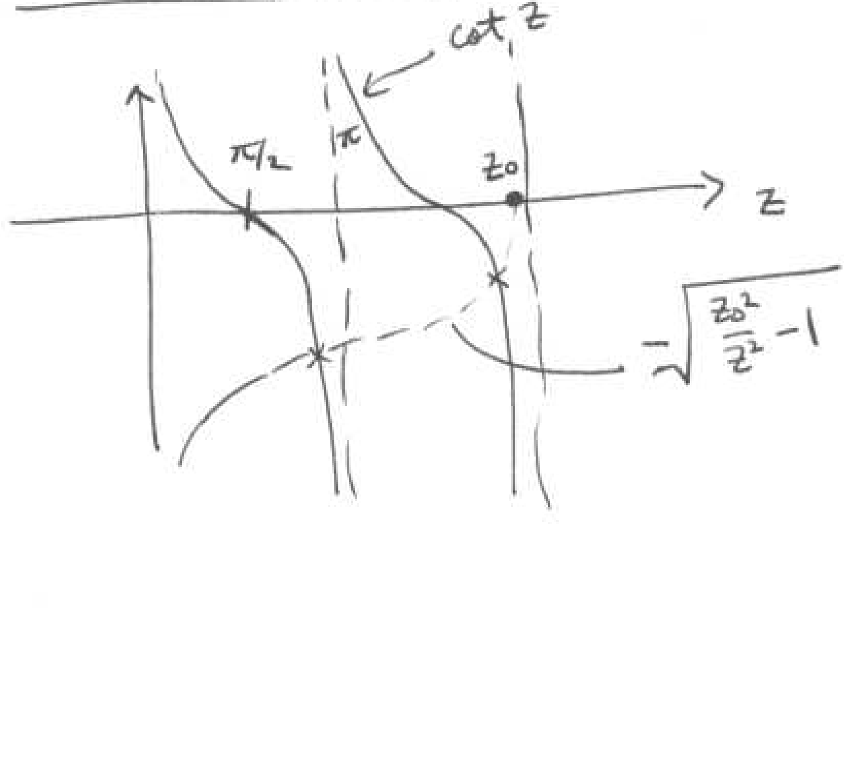
Then $K^2 + k^2 = \frac{-2mE}{\hbar^2} + \frac{2m(E+V_0)}{\hbar^2} = \frac{2mV_0}{\hbar^2} = \frac{z_0^2}{a^2}$.

$\Rightarrow \quad Ka = \sqrt{z_0^2 - k^2a^2} = \sqrt{z_0^2 - z^2}$.

Thus $\frac{K}{k} = \sqrt{\frac{z_0^2}{z^2} - 1}$.

So we have to solve $\underline{\cot z = -\sqrt{\frac{z_0^2}{z^2} - 1}}$.

Look at a plot



There will be no solution (no intersection) if $z_0 < \pi/2$.

Thus $\frac{a}{\hbar}\sqrt{2mV_0} < \pi/2$

$\Rightarrow$ No bound state for.

$$\boxed{V_0 < \frac{\pi^2\hbar^2}{8ma^2}}$$