Problem 1. (20 points)

There are questions on the back of this page.

(a) [4 points] The radial component of momentum in spherical coordinates is given by

$$\hat{p}_r \psi = \frac{\hbar}{\mathrm{i}} \frac{1}{r} \frac{\partial}{\partial r}(r\psi).$$

Use this to calculate the commutator $[\hat{r}, \hat{p}_r]$.

(b) [4 points] An electron in a hydrogen atom is in the $\ell = 6$ state. What is the minimum energy it could have?

Problem 1 continued:
An electron starts in the spin-up state, $\chi(0) = \chi_+$, at $t = 0$.

(c) [3 points] If $\hat{S}_x$ were measured what is the probability of getting the value $\hbar/2$?

(d) [4 points] If instead of making a measurement of the original spin we instead turn on a constant magnetic field in the z-direction, $\boldsymbol{B} = B_0\hat{k}$, what is $\chi(t)$?

(e) [5 points] If $\hat{S}_x$ were measured now measured at time t what is the probability of getting the value $\hbar/2$? Explain why this answer makes physical sense.

Problem 2. (30 points)

The state of an electron in a hydrogen atom is given by

$$\psi(\theta, \phi) = A\left[3\sin\theta\cos\theta\, e^{i\phi}\chi_+ - 2(1-\cos^2\theta)e^{2i\phi}\chi_-\right].$$

[***Note*: There are questions on the back of this page.**]

(a) [6 points] Write the state in terms of spherical harmonics and normalize it. [*Hint*: The coefficients in front of the two terms will be simple fractions.]

(b) [4 points] If $\hat{L}_z$ were measured what values would be found and with what probabilities?

(c) [4 points] If $\hat{S}_z$ were measured what values would be found and with what probabilities?

Problem 2 continued:

The total angular momentum of this state is given by $\hat{\boldsymbol{J}} = \hat{\boldsymbol{L}} + \hat{\boldsymbol{S}}$.

(d) [4 points] What are the allowed values for the quantum numbers j and m_j for this state?

(e) [8 points] Expand this state in terms of the $\{|j, m_j\rangle\}$ basis. [*Hint*: Expand each term and combine the results. Your answer should already be normalized.]

(f) [4 points] If $\hat{J}^2$ were measured what values would be found and with what probabilities?

Problem 3. (15 points)

Two identical spin- $^1/_2$ particles, labeled 1 and 2, are separated by a distance, $\boldsymbol{a} = a\hat{z}$, and interact through a dipole coupling given by the Hamiltonian

$$\hat{H} = \frac{1}{a^3}\left[\boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_2 - \frac{3}{a^2}(\boldsymbol{\mu}_1 \cdot \boldsymbol{a})(\boldsymbol{\mu}_2 \cdot \boldsymbol{a})\right],$$

where $\boldsymbol{\mu}_j$ is the dipole moment of particle j. [***Note*: There is a question on the back of this page.**]

(a) [6 points] Write the Hamiltonian in terms of $\hat{\boldsymbol{S}}^{(1)}$ and $\hat{\boldsymbol{S}}^{(2)}$.

(b) [4 points] Rewrite the Hamiltonian in terms of the total spin angular momentum operators, $\hat{S}^2$, $\hat{S}_z$, and in terms of $\left[\hat{S}^{(j)}\right]^2$ and $\hat{S}_z^{(j)}$ where $\hat{\boldsymbol{S}} = \hat{\boldsymbol{S}}^{(1)} + \hat{\boldsymbol{S}}^{(2)}$.

Problem 3 continued:

(c) [5 points] For the two spin-$^1/_2$ particles find the energy of the total spin state $|0, 0\rangle$.