**Problem 1.** (28 points)

Clearly explain your answers to the following questions. **There are questions on the back of this page.**

**(a)** [6 points] For some operator, $\hat{B}$, let $\hat{B}|\psi_n\rangle = 2^n|\psi_n\rangle$. Given the state,

$$|\psi\rangle = A\left(\frac{1}{\sqrt{2}}|\psi_0\rangle + \frac{1}{\sqrt{5}}|\psi_1\rangle + \frac{1}{\sqrt{10}}|\psi_2\rangle\right)$$

Find $A$ and calculate $\langle\hat{B}\rangle$.

**(b)** [4 points] In a quantum mechanics book I found the following statement in a problem: Consider three observables, $\hat{A}$, $\hat{B}$, and $\hat{C}$ that satisfy $[\hat{B}, \hat{C}] = \hat{A}$. Explain why these three operators *cannot* all be observables.

**(c)** [6 points] What the book referenced in the previous problem meant to say was that for these operators suppose that $[\hat{B}, \hat{C}] = \mathrm{i}\hat{A}$ and $[\hat{A}, \hat{C}] = \mathrm{i}\hat{B}$. Now calculate the uncertainty relation for $\sigma_{AB}\sigma_C$.

**Problem 1 continued:**

**(d)** [8 points] Find the eigenfunctions, $\psi(x)$, of the ladder operator, $\hat{a}_+$, defined for the simple harmonic oscillator. What are the allowed eigenvalues? [*Hint*: You should find a first order differential equation that is easy to integrate.]

**(e)** [4 points] Are the eigenfunctions you found in the previous part also eigenfunctions of $\hat{a}_-$? [*Note*: You can either explain whether they can or cannot be or show the result by explicit calculation.]

**Problem 2.** (32 points)
The states $\{|1\rangle, |2\rangle, |3\rangle\}$ form a basis in a three dimensional Hilbert space. Consider the Hamiltonian

$$\hat{H} = |1\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 1| - |2\rangle\langle 3| - |3\rangle\langle 2| + |3\rangle\langle 3|.$$

[*Note*: **There are questions on the back of this page.**]

**(a)** [4 points] Write the Hamiltonian as a matrix and verify that it is Hermitian.

**(b)** [10 points] Find the eigenvalues, $\{E_i\}$, and normalized eigenvectors, $\{|e_i\rangle\}$ for $\hat{H}$. Make sure it is clear which eigenvalue corresponds to which eigenvector.

**Problem 2 continued:**

**(c)** [6 points] Suppose the system starts in the state

$$|\mathscr{S}(0)\rangle = \begin{pmatrix} 0 \\ \mathrm{i} \\ 0 \end{pmatrix}.$$

Write $|\mathscr{S}(0)\rangle$ in the original basis $\{|1\rangle, |2\rangle, |3\rangle\}$ and in the basis of the eigenstates of the Hamiltonian, $\{|e_i\rangle\}$

**(d)** [8 points] Write $|\mathscr{S}(t)\rangle$ in both bases.

**(e)** [4 points] Calculate the probability of measuring the system in the state $|\mathscr{S}(0)\rangle$ as a function of time. Verify that you get the correct answer at $t = 0$.