Problem 1. (16 points)

[Note: There are questions on the back of this page.]

(a) [6 points] Describe in words what each of the quantities below stands for and the physics of this equation:

\[
\frac{dW}{dt} = -\frac{\partial u_{EM}}{\partial t} - \nabla \cdot \mathbf{S}.
\]

This describes the "work-energy" theorem for EM fields. 

\(W\) is the work done by the EM fields (work density really), 
\(u_{EM}\) is the energy density in EM fields, 
\(S\) is the Poynting vector and \(\nabla \cdot \mathbf{S}\) describes the flow of energy.

(b) [4 points] In general, good conductors are also good reflectors of visible light. Explain why this is the case.

The skin depth for a good conductor is very small. This means an EM wave does not penetrate very far into the conductor so that most of the wave is reflected.
Problem 1 continued:

(c) [4 points] A laser beam can be treated as an electromagnetic wave. Consider a laser beam with $20 \times 10^9$ W of power collimated into a 1 mm radius circular beam. Calculate the amplitude of the electric field of the laser beam in units of volts per meter. [Hint: What does the intensity of an electromagnetic wave tell us? For convenience recall that in SI units $\varepsilon_0 = 8.85 \times 10^{-12} \text{ C}^2/\text{Nm}^2$, $\mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2$, and $c = 3 \times 10^8 \text{ m/s}$.]

Recall that $I = \frac{1}{2} \varepsilon_0 E_0^2$ is the average power per unit area.

Thus $I = \frac{P}{A}$ with $A = \pi a^2$.

So $E_0 = \sqrt{\frac{2I}{\varepsilon_0 c}} = \sqrt{\frac{2P}{\pi \varepsilon_0 c \varepsilon_0 c}}$

Plugging in numbers we find $E_0 = 2.19 \times 10^9 \frac{\text{V}}{\text{m}}$

(d) [2 points] For the laser beam from the previous part, calculate the amplitude of the magnetic field in units of Tesla. [Note: To get some idea of scale note that the magnetic field of the Earth is about half a Gauss, or equivalently about $5 \times 10^{-5}$ T!]

$B_0 = \frac{E_0}{c} = 7.30 \text{T}$

Notice $B_0 \gg B_{\text{Earth}}$. 
Problem 2. (22 points)

Consider a solenoid of radius, \( a \), and length, \( L \). Assume that the \( L \gg a \) so we can treat the solenoid as ideal. Let \( B_0 \) be the magnetic field inside the solenoid. [Note: There are questions on the back of this page.]

(a) [4 points] Calculate the total energy stored in the electromagnetic field inside the solenoid.

\[ \text{Inside the solenoid} \quad B = B_0 \hat{z} \quad \text{for} \quad \rho \leq a \]

Thus, \( U_{EM} = \frac{1}{2 \mu_0} B_0^2 = \frac{B_0^2}{2 \mu_0} \).

\[ U_{EM} = \int_{\text{solenoid}} U_{EM} d\mathbf{r} = \frac{B_0^2}{2 \mu_0} \left( \pi a^2 L \right) \]

\[ \Rightarrow U_{EM} = \frac{\pi a^2 L B_0^2}{2 \mu_0} \]

(b) [6 points] Calculate all the components of the Maxwell stress tensor, \( T_{ij} \).

Since \( \mathbf{E} = 0 \) the Maxwell stress tensor is given by

\[ T_{ij} = \frac{1}{\mu_0} \left( B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right) \]

where \( B_z = B_0 \), \( B_x = B_y = 0 \).

Thus, \( T_{xx} = T_{yy} = T_{yx} = T_{xy} = 0 \)

\[ T_{xx} = T_{yy} = -\frac{1}{2 \mu_0} B_0^2 = -\frac{B_0^2}{2 \mu_0} \]

\[ T_{zz} = \frac{1}{\mu_0} \left( B_0^2 - \frac{1}{2} B^2 \right) = \frac{B_0^2}{2 \mu_0} \]

Thus, \( T = \frac{B_0^2}{2 \mu_0} \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \)
Problem 2 continued:

(c) [6 points] Calculate the force on the “top” end cap of the solenoid. Does the force lead to compression or expansion of the solenoid? Clearly justify your answer.

On the top cap $d\alpha = \rho \, dp \, dy \, \hat{z}$. \[ \mathbf{F} = \int d\mathbf{F} = \int \mathbf{E} \cdot d\mathbf{F}. \]

$\Rightarrow \mathbf{F}_z = \int T_z \, dp \, dy = \int T_z \, \rho \, dp \, dy$

Thus we only have a $z$ component. [This is expected from symmetry.]

So $\mathbf{F} = F_z \, \hat{z}$ where

$$F_z = \int T_z \, \rho \, dp \, dy = \int_0^{2\pi} d\phi \int_0^a \rho \, d\rho \, \frac{B_0^2}{2\mu_0}$$

$\Rightarrow \boxed{F_z = \frac{\pi a^2 B_0^2}{2\mu_0}}$

Since $F_z > 0 \Rightarrow$ solenoid expands.

(d) [4 points] Calculate the pressure exerted on the cylindrical walls of the solenoid. Is this pressure inward or outward? Clearly justify your answer. [Hint: No calculation is actually required, you can just write down the answer.]

The diagonal elements of the stress tensor are the pressures.

Thus the pressure on the walls is just $-\frac{B_0^2}{2\mu_0}$

Since pressure is negative this is compression.
Problem 3. (22 points)
The electric field of a plane wave in vacuum may be written in cylindrical coordinates as

\[ E(r, t) = \frac{E_0}{s} \cos(kz - \omega t) \hat{s}, \]

where \( E_0 \) is a constant. [Note: There are questions on the back of this page.]

(a) [4 points] Derive the magnetic field, \( B(r, t) \), for this wave.

In vacuum \( \overrightarrow{B} = \frac{1}{c} \hat{k} \times \overrightarrow{E} \). Here \( \hat{k} = \hat{z} \), so

\[ \overrightarrow{B} = \frac{E_0}{cs} \cos(kz - \omega t) \hat{\phi} \]

Thus

\[ \overrightarrow{B}(r, t) = \frac{E_0}{cs} \cos(kz - \omega t) \hat{\phi} \]

(b) [8 points] Verify that the Maxwell equations \( \nabla \cdot E = 0, \nabla \cdot B = 0, \nabla \times E = -\partial B/\partial t \) are satisfied for this cylindrical wave. [Note: We could also verify the final Maxwell equation, but that is more tedious algebra.]

In cylindrical coordinates

\[ \nabla \cdot \overrightarrow{E} = \frac{1}{s} \frac{\partial}{\partial s} (sE_s) + \frac{1}{r} \frac{\partial}{\partial r} (rE_r) + \frac{1}{r} \frac{\partial E_\phi}{\partial \phi} = 0 \]

Thus for \( \overrightarrow{E} \):

\[ \nabla \cdot \overrightarrow{E} = \frac{1}{s} \frac{\partial}{\partial s} (sE_s) = 0 \quad \text{ind. of } s \]

and for \( \overrightarrow{B} \):

\[ \nabla \cdot \overrightarrow{B} = \frac{1}{s} \frac{\partial}{\partial s} \left( \frac{E_0}{cs} \cos(kz - \omega t) \right) = 0 \quad \text{ind. of } s \]

For the curl, since \( \overrightarrow{E} \) only has \( \hat{s} \) component we have

\[ \nabla \times \overrightarrow{E} = \frac{\partial E_\phi}{\partial r} \hat{\phi} - \frac{1}{r} \frac{\partial E_r}{\partial \phi} \hat{z}, \quad \text{but } \frac{\partial E_s}{\partial \phi} = 0 \quad \text{and } \frac{\partial E_\phi}{\partial s} = -\frac{kE_0}{s} \sin(kz - \omega t) \]

Also,\n
\[ \frac{\partial \overrightarrow{B}}{\partial t} = \frac{\omega E_0}{cs} \sin(kz - \omega t) \hat{\phi}, \quad \text{Thus } \nabla \times \overrightarrow{E} = -\frac{kE_0}{s} \sin(kz - \omega t) \hat{\phi} = -\frac{\partial \overrightarrow{B}}{\partial t} \]

as required since \( \hat{k} = \frac{\omega}{c}. \)
Problem 3 continued:

(c) [4 points] Calculate the Poynting vector, $\mathbf{S}$, for this wave.

$$\mathbf{S} = \mathbf{E} \times \mathbf{B} = \frac{1}{\mu_0} \frac{E_0^2}{c s^2} \cos^2(kz - \omega t) \hat{z} \times \hat{y} = \frac{\hat{z}}{2}.$$ 

$$S_0 = \frac{E_0^2}{\mu_0 c s^2} \cos^2(kz - \omega t) \hat{z} = \frac{E_0 c}{s^2} \frac{E_0^2}{s^2} \cos^2(kz - \omega t) \hat{z}.$$ 

Since $\frac{1}{\mu_0 c} = \frac{c}{\mu_0 c^2} = \frac{E_0 \mu_0 c}{\mu_0} = E_0 c.$

(d) [4 points] Calculate the average of the Poynting vector over one period, $\mathbf{I} = \langle \mathbf{S} \rangle$. This gives the intensity as a function of distance from the source. Does this behave as expected? [Note: All the integrals required for this calculation are trivial!]

$$\mathbf{I} = \langle \mathbf{S} \rangle = \frac{E_0^2}{\mu_0 c s^2} \langle \cos^2(kz - \omega t) \rangle \hat{z}$$

$$\mathbf{I} = \frac{E_0^2}{2 \mu_0 c s^2} \hat{z} = \frac{E_0 c E_0^2}{2 s^2} \hat{z}$$

Notice that $\mathbf{I}$ points in the direction of propagation for the wave. Also, it falls off as $\frac{1}{s^2}$ as we might expect.

(e) [2 points] Calculate the total power transmitted through a circular annulus with $a \leq s \leq b$ aligned parallel to the $xy$-plane.

$$P = \int_{\mathbf{I}} d\mathbf{a}.$$ Here $d\mathbf{a} = s ds d\phi \hat{z}$

$$P = \int_0^{2\pi} d\phi \int_a^b s ds \frac{E_0^2}{2 \mu_0 c s^2} = \frac{2\pi E_0^2}{2 \mu_0 c} \int_a^b \frac{ds}{s}$$

$$P = \frac{\pi E_0^2}{\mu_0 c} \ln \left( \frac{b}{a} \right) = \pi E_0 c E_0^2 \ln \left( \frac{b}{a} \right)$$