14) P10.3

(a) \( V(\vec{r}, t) = 0 \), \( \vec{A}(\vec{r}, t) = -\frac{1}{4\pi\epsilon_0} \frac{q t}{r^2} \hat{r} \)

\[ \vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} = \frac{1}{4\pi\epsilon_0} \frac{q t}{r^2} \hat{r} \implies \]

\[ \vec{B} = \vec{\nabla} \times \vec{A} = 0 \]

These fields clearly describe a stationary point charge, \( q \).

We can write them mathematically as \( \rho(\vec{r}, t) = q \delta(r) \)

\( \frac{\partial \vec{A}}{\partial t} = 0 \)

(b) Let \( \Lambda = -\frac{q t}{4\pi\epsilon_0 r} \)

\[ \vec{V}' = \vec{V} - \frac{\partial \Lambda}{\partial \hat{r}} = \frac{1}{4\pi\epsilon_0 r} \implies \]

\[ \vec{V}'(\vec{r}, t) = \frac{1}{4\pi\epsilon_0 r} \]

\[ \vec{A}' = \vec{A} - \vec{\nabla} \Lambda. \]

In spherical coordinates, \( \vec{\nabla} \Lambda = \frac{2}{r^2} \hat{r} \) since \( \Lambda(r) \).

so \( \vec{\nabla} \Lambda = \frac{q t}{4\pi\epsilon_0 r^2} \hat{r} \)

\[ \implies \vec{A}'(\vec{r}, t) = 0 \]

These potentials are the more natural choice for a static point charge. In fact, we have used them many times! Even so, there is nothing wrong with the choice in (a), they just look strange.
2) In this problem we want to see how we find the function \( \lambda(\vec{r},t) \) to satisfy gauge conditions.

Suppose we have \( V(\vec{r},t) \) and \( \tilde{A}(\vec{r},t) \) as known functions and we wish to find new potentials \( V'(\vec{r},t) \) and \( \tilde{A}'(\vec{r},t) \) using the freedom

\[ \tilde{A}' = \tilde{A} + \vec{\nabla} \lambda, \quad \text{and} \quad V' = V - \frac{\partial \lambda}{\partial t}. \]

(i) \( \vec{\nabla} \cdot \tilde{A} = 0 \): Suppose we start with \( \vec{\nabla} \cdot \tilde{A} \neq 0 \), we require

\[ \vec{\nabla} \cdot \tilde{A}' = 0 = \vec{\nabla} \cdot \tilde{A} + \vec{\nabla} \lambda, \quad (\vec{\nabla} \lambda) = \vec{\nabla} \cdot \tilde{A} + \nabla^2 \lambda \]

\[ \Rightarrow \, \nabla^2 \lambda = -\vec{\nabla} \cdot \tilde{A}, \quad \text{The right hand side is a known function, so this is just the Poisson equation which we know how to solve. This is sufficient, we have reduced the problem to one with a known solution, so we know we can satisfy the gauge condition using the gauge freedom.} \]

Explicitly, \( \lambda = \frac{1}{4\pi} \int \frac{\vec{\nabla} \cdot \tilde{A}}{r^2} \, dt' \)

(ii) \( \vec{\nabla} \cdot \tilde{A} = -\mu_0 \delta \frac{\partial V}{\partial t} \): We proceed as in (i) using the gauge freedom:

\[ \Rightarrow \, \vec{\nabla} \cdot \tilde{A} + \nabla^2 \lambda = -\mu_0 \delta \frac{\partial V}{\partial t} + \mu_0 \delta \frac{\partial^2 V}{\partial t^2} \]

\[ = \left( \nabla^2 - \mu_0 \delta \frac{\partial^2}{\partial t^2} \right) \lambda = -\vec{\nabla} \cdot \tilde{A} - \mu_0 \delta \frac{\partial V}{\partial t} \]

\[ \Rightarrow \, \nabla^2 \lambda = -\delta \]

This is just the inhomogeneous wave equation.

We know it has a solution so we are done!

[We could write out solutions but this is not necessary.]
2) can’t

(iii) \( V = 0 \): This can be achieved by choosing \( V = \frac{dI}{dt} \)

Again, this is not sufficient. Here the general result is

\[ \mathcal{A}(\mathbf{r}, t) = \int_0^t V(\mathbf{r}, t'') dt'' \]

(iv) \( \dot{\mathbf{A}} = 0 \): In general we cannot do this!

If we could transform to \( \dot{\mathbf{A}} = 0 \) this would mean \( \dot{\mathbf{B}} = \nabla \times \dot{\mathbf{A}} = 0 \) is always true. But we know we can have \( \mathbf{B} \) fields, so we cannot always perform this gauge transformations. In other words, \( \dot{\mathbf{A}} = 0 \) is not a valid gauge choice.
(36) (i) Let $I(t) = k(t)$ for $t > 0$

As $t \to \infty$, we have $I \to \infty$ and constantly growing, so we expect both $\mathbf{B} \to \infty$ and $\mathbf{E} \to \infty$. This is not such an interesting case, but still worth thinking about.

(ii) We follow the same procedure as in class, now with $I(\mathbf{r}, t) = k(t) \gamma (t - \frac{\mathbf{r}}{c})$

and $\mathcal{N} = \sqrt{s^2 + z'^2}$.

We still have $V(\mathbf{r}, t) = 0$ and calculate

\[
\overline{A}(\mathbf{r}, t) = \frac{M_0}{4\pi} \gamma \left(2\right) \int_0^{\sqrt{ct^2 - s^2}} \frac{k(t - \frac{1}{c}\sqrt{s^2 + z'^2})}{\sqrt{s^2 + z'^2}} \, dz'
\]

\[
= \frac{M_0 k}{2\pi c} \left[ t \ln \left( \frac{ct + \sqrt{(ct)^2 - s^2}}{s} \right) - \frac{1}{c} \sqrt{(ct)^2 - s^2} \right]
\]

The first term is similar to that found in example 10.2, so we can use results from our derivation in class.

To find $\frac{\partial \overline{A}}{\partial t}$, there will be 3 terms. One was calculated in class:

Term with $\frac{\partial}{\partial t} \left[ \ln(\cdot) \right]$ \[ \to - \frac{M_0 k}{2\pi c} \frac{ct^2}{\sqrt{(ct)^2 - s^2}} \]

From the last term, we get $- \frac{M_0 k}{2\pi c} \frac{ct^2}{\sqrt{(ct)^2 - s^2}}$. Thus

\[
\mathbf{E}(\mathbf{r}, t) = - \frac{M_0 k}{2\pi c} \gamma \ln \left( \frac{ct + \sqrt{(ct)^2 - s^2}}{s} \right)
\]

\[
\mathbf{B} = \nabla \times \mathbf{A} = - \frac{M_0 k}{2\pi c} \gamma \nabla \ln \left( \frac{ct + \sqrt{(ct)^2 - s^2}}{s} \right)
\]

\[
\mathbf{B} = \frac{M_0 k}{2\pi c} \gamma \left[ \frac{ct^2}{s\sqrt{(ct)^2 - s^2}} - \frac{s}{c \sqrt{(ct)^2 - s^2}} \right] = \frac{M_0 k}{2\pi c s} \gamma \sqrt{(ct)^2 - s^2}
\]

\[
\mathbf{B}(\mathbf{r}, t) = \frac{M_0 k}{2\pi c s} \gamma \sqrt{(ct)^2 - s^2}
\]
(i) Now consider $I(t) = q_0 \delta(t)$.

As $t \to \infty$ we have $I(t) \to 0$ so we expect $E = \mathbf{B} = 0$.

(ii) $I(t) = q_0 \delta(t-t') = q_0 \delta(t-\frac{r}{c}) = q_0 \delta(t-\frac{1}{c}\sqrt{s'^2+z'^2})$.

Again $V(\mathbf{r};t) = 0$ and

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi} \int_0^\infty \frac{q_0 \delta(t-\frac{1}{c}\sqrt{s'^2+z'^2})}{\sqrt{s^2+z'^2}} \, dz'.$$

To integrate the $\delta$-fun we need to be careful.

Let $U = \frac{1}{c} \sqrt{z'^2+s'^2}$, then $du = \frac{1}{c} \sqrt{z'^2+s'^2} \, dz' \Rightarrow \frac{dz'}{\sqrt{z'^2+s'^2}} = \frac{c}{\sqrt{c^2u^2-s'^2}}$.

With this change $U = \frac{z'}{c}$ with $z' \to 0$, $U = \infty$ when $z' = \infty$, so

$$\mathbf{A} = \frac{\mu_0}{2\pi} \int_{\frac{1}{c}}^\infty \frac{c \delta(t-u)}{\sqrt{c^2u^2-s'^2}} \, du = \frac{\mu_0 q_0 c}{2\pi} \frac{1}{\sqrt{(ct)^2-s'^2}}$$

so $t > \frac{1}{c}$.

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = \frac{\mu_0 q_0 c}{2\pi} \left( \frac{1}{(ct)^2-s'^2} \right)^{\frac{3}{2}} \frac{1}{c^3}$$

$$\mathbf{E}(\mathbf{r};t) = \frac{\mu_0 q_0 c^3 t}{2\pi \left[ (ct)^2-s'^2 \right]^{\frac{3}{2}}}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial s} \hat{t} = -\frac{\mu_0 q_0 c}{2\pi} \left( \frac{1}{(ct)^2-s'^2} \right)^{\frac{3}{2}} (\frac{1}{c}) \hat{t}$$

$$\mathbf{B}(\mathbf{r};t) = -\frac{\mu_0 q_0 c s}{2\pi \left[ (ct)^2-s'^2 \right]^{\frac{3}{2}}}$$

Notice both $\to 0$ as $t \to \infty$. 