1. An alternative formulation of the relativistic velocity addition formula is:

\[
\left( \frac{c-u}{c+u} \right) = \left( \frac{c-u'}{c+u'} \right) \left( \frac{c-v}{c+v} \right)
\]

(a) show that in our usual frames \(S'\), \(S\), this is equivalent to

\[u = \frac{u' + v}{1 + \frac{u'v}{c^2}}\]

Note: Just algebra.

(b) By considering this new formulation (using different notation to help deal w/multiple reference frames) we can see that it can be used to chain velocity additions together:

\[
\left( \frac{c-V_{CA}}{c+V_{CA}} \right) = \left( \frac{c-V_{CB}}{c+V_{CB}} \right) \left( \frac{c-V_{BA}}{c+V_{BA}} \right)
\]

Represented in our frames we have:

\[S' \rightarrow \vec{V}_{BA} \rightarrow \vec{V}_{CB} \rightarrow \rightarrow S \rightarrow \vec{V}_{CA}\]

So we can get from \(A \rightarrow C\) by compounding \(A \rightarrow B \rightarrow C\). Generalize this to solve the following: Suppose in \(S\) there is a rocket moving at \(V = 5c/6\), which emits a rocket moving at \(V = 2c/3\) with respect to its frame, which emits a rocket moving at \(V = c/2\) with respect to its frame, which emits a rocket moving at \(V = c/3\) with respect to its frame, which emits a rocket moving at \(V = c/6\) with respect to its frame. (All are moving to the right.) How fast is the final rocket moving in \(S\)? Note: Doing this using the conventional velocity addition formula is not recommended!

2. Do a version of the “barn problem” (Harris 2.24 is an example) schematically as a spacetime diagram. Use graph paper for this problem, use a ruler to draw straight lines on it. It’s also useful, but not required, to use different colors (makes things easier to read). What we are trying to show here is that length contraction is symmetrical: in a given frame sticks at rest in other frames oriented along the direction of relative motion will be contracted – this works in both directions.

If you need a reminder on constructing these diagrams you can review the spacetime diagram that we discussed in class for the exploding planet problem:
A rough sketch of what follows is recommended before doing the careful drawing.

(a) On graph paper draw two vertical lines spaced a reasonable distance apart to represent a stick in some reference frame. Each line represents a world line for an end of the stick. In this picture a horizontal line represents simultaneous events (given that a vertical one represents constant location): draw a horizontal line and mark the two intersection points with the vertical lines A and B – to represent, respectively, the left and right ends of the stick at some moment in time in this frame.

(b) Draw the null line – represents the world line of things moving to the right at the speed of light in this (or any) reference frame – extending from the point marked A.

(c) Draw the time and space axes for a frame moving rightwards relative to the first such that they intersect at the left end of the stick (point A) at rest in the first frame. Choose a speed that is fast enough (but not too fast) that we’ll be able to see the effect.

(d) Draw world lines for the ends of a stick at rest in this second frame. You will have to choose the length of this stick carefully in order to demonstrate the effect – make it a bit shorter, but not too much, than the stick in the first reference frame (as measured in the first reference frame), but longer in its own frame. Mark the ends of this stick in its rest frame A’ and B’ at some moment in time (in that second frame).

(e) Mark the end of the stick at rest in the second frame as measured in the first frame as C – A, B, and C should lie on a line of simultaneity in this frame, so that we may compare the 2 lengths in this frame.

(f) Mark the end of the stick at rest in the first frame as measured in the second frame as C’ – A’, B’, and C’ should lie on a line of simultaneity in this frame, so that we may compare the 2 lengths in this frame.

(g) Comment on the results, by comparing lengths in a single frame, for each frame.

3.

(a) Given the rule for multiplying 4-vectors (the notation below is different than Harris’s section 2.10 – but the results are equivalent. We will use this notation quite a bit in lecture.):

\[ \tilde{a} \cdot \tilde{b} = (a_0, \vec{a}) \cdot (b_0, \vec{b}) = a_0b_0 - \vec{a} \cdot \vec{b} \]

And that \( \tilde{x} \equiv (ct, \vec{x}) \), show that \( \tilde{x}^2 \) is an invariant. Note: Do this by transforming \( \tilde{x} \) to another frame using the Lorentz Transformations.

(b) The four-momentum \( \tilde{p} \equiv (E/c, \vec{p}) \). For the following find \( \tilde{p} \) and \( \tilde{p}^2 \). Express your answers numerically with appropriate units. You may assume these particles are moving in a specific direction, say \( \hat{x} \), so it’s only necessary to give one component of the momentum, thus you may write \( \tilde{p} \) answers in the form \((20 \text{ MeV}/c, 10 \text{ MeV}/c)\) rather than \((20 \text{ MeV}/c, 10 \text{ MeV}/c, 0, 0)\).

- a pion \( m_\pi = 140 \text{ MeV}/c^2 \), KE = 100 MeV
• a photon (massless) with $E = 10$ MeV
• a proton $m_p = 938$ Mev/$c^2$, $v = \sqrt{3}/2$ c

4. Harris 2.76
But instead of using the relativistic velocity transform to transform to the new frame in part (b), use the Lorentz Transform (trust me, it’s easier)

$$
p' = \gamma (p - \frac{vE}{c^2})
$$
$$
E' = \gamma (E - vp)
$$

and continue with the rest of the problem as written, except find the momenta in the new frame instead of the velocities (typically particle velocities are of little concern). Note: You won’t need to transform E, I listed it for completeness.

5. Show that an electron and a positron (an anti-electron, it has the same mass and opposite charge), cannot annihilate into a single photon. Hint: Check energy and momentum conservation.

6. Harris 2.93