Phys 221 Sample Problems III

These are problems which are similar to some that have been given on previous exams in this course. Solutions will not be provided, but feel free to ask questions on the bulletin board – I may or may not have worked out the answers. You are encouraged to work together on these.

Homework problems and problems like them from the book are good for review.

1. An infinite well has the potential

\[ U(x) = \begin{cases} 
\infty & : \ x < 0 \\
0 & : \ 0 < x < \frac{L}{2} \\
U_0 & : \ \frac{L}{2} < x < L \\
\infty & : \ x > L 
\end{cases} \]

(a) Sketch a plausible ground state wave function for this potential for the case \( E < U_0 \).

(b) Write down general solutions to the Schrödinger equation in the 4 regions

\[ I : x < 0 \quad II : 0 < x < \frac{L}{2} \quad III : \frac{L}{2} < x < L \quad IV : x > L \]

Define any parameters which are not amplitudes.

(c) Apply boundary conditions to these solutions to eliminate as many pieces and as many coefficients as possible.

(d) Find the transcendental equation which would yield the energy eigenvalues.

2. Consider the potential

\[ U(x) = \begin{cases} 
\infty & : \ x < 0 \\
-b/x & : \ x > 0 
\end{cases} \]

(a) Show that \( \psi(x) = Axe^{-ax} \) for \( x \geq 0 \) is a solution to the Schrödinger equation for this potential, determining any conditions (e.g. relations between parameters) that must be satisfied.

(b) Find \( E \) in terms of \( h, m \) and \( b \).

(c) What is \( \psi(x) \) for \( x \leq 0 \)?

(d) Discuss the smoothness conditions on \( \psi(x) \) at \( x = 0 \).
3. A finite well has the potential

\[ U(x) = \begin{cases} 
0 & : x < 0 \\
-U_0 & : 0 < x < L \\
0 & : x > L 
\end{cases} \]

where \( U_0 > 0 \).

Considering particles scattering from \( x < 0 \).

(a) Draw a graph of the potential and write down general solutions to the Schrödinger equation in the 3 regions

I : \( x < 0 \) II : \( 0 < x < L \) III : \( x > L \)

Define any parameters which are not coefficients (amplitudes).

(b) Apply boundary/initial conditions to these solutions to eliminate and relate as many coefficients as possible. You do not need to solve the boundary condition equations.

(c) If this problem was worked out fully we would find that the reflection coefficient is

\[ R = \frac{\sin^2 \left( \sqrt{2m(E + U_0)L/h} \right)}{\sin^2 \left( \sqrt{2m(E + U_0)L/h} \right) + 4(E/U_0)((E/U_0) + 1)} \]

Find an expression for the energies that are transmitted with 100% efficiency.

4. An well has the potential

\[ U(x) = \begin{cases} 
\infty & : x < 0 \\
0 & : 0 < x < \frac{3L}{4} \\
U_0 & : \frac{3L}{4} < x < L \\
0 & : x > L 
\end{cases} \]

take \( U_0 = 12 \text{ eV}, \ E = 5.5 \text{ eV}, \ L = 1.75 \text{ nm} \), assuming that the particle is an electron and is initially in the well – explain why this is not a stable state. Estimate its lifetime.
5. For the Hydrogen atom

\[ U(r) = \frac{-e^2}{4\pi\epsilon_0 r} \]

(a) Find the classical turning point in terms of the Bohr radius, \( a_0 \).

Suppose an electron is in the state \( \psi_{2,1,-1} \).

(b) Write down the normalized wave-function.

(c) Find \( \langle L^2 \rangle \)

(d) Find \( \langle L_z \rangle \)

(e) Find the probability that the \( \psi_{2,1,-1} \) electron is in the classically forbidden region.

Note: \( \int x^n e^{ax} dx = \frac{e^{ax}}{a} \left( x^n - \frac{n x^{n-1}}{a} + \frac{n(n-1)x^{n-2}}{a^2} - \ldots + \frac{(-1)^n n!}{a^n} \right) \) for positive integer \( n \).

6. Consider a 1D infinite well which contains 2 non-interacting spin \( \frac{1}{2} \) identical fermions. The eigenfunctions of the 1D well are:

\[ \psi_n = \sqrt{\frac{2}{L}} \sin \left( \frac{n\pi x}{L} \right) \]

Find the lowest energy state for each spin state given below – just list the quantum numbers for \( E_{n_1, n_2} \). And write down the corresponding spatial part of the wavefunction (unnormalized is fine):

(a) The spin state is \( \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \)

(b) The spin state is \( \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \)

(c) For which of these states is the probability higher that both particles will be found on the same side of the box? Write down the expressions that would allow you to calculate the probabilities in each case – but don’t evaluate the integrals unless you really, really want to, instead explain in words why they’d differ (or not).
7.
For a 3D quantum gas,

\[ D(E) = \frac{(2s + 1)m^{3/2}V}{\pi^2 \hbar^3 \sqrt{2}} E^{1/2} \]

Suppose the particles are electrons.

(a) If the gas is at zero temperature: find \( \bar{E} \), the average energy, in terms of \( E_F \), the Fermi energy.

(b) For \( k_B T \gg E_F(0) \), where \( E_F(0) \) is the zero-temperature Fermi Energy: find \( \bar{E} \) (as a function of \( T \)).