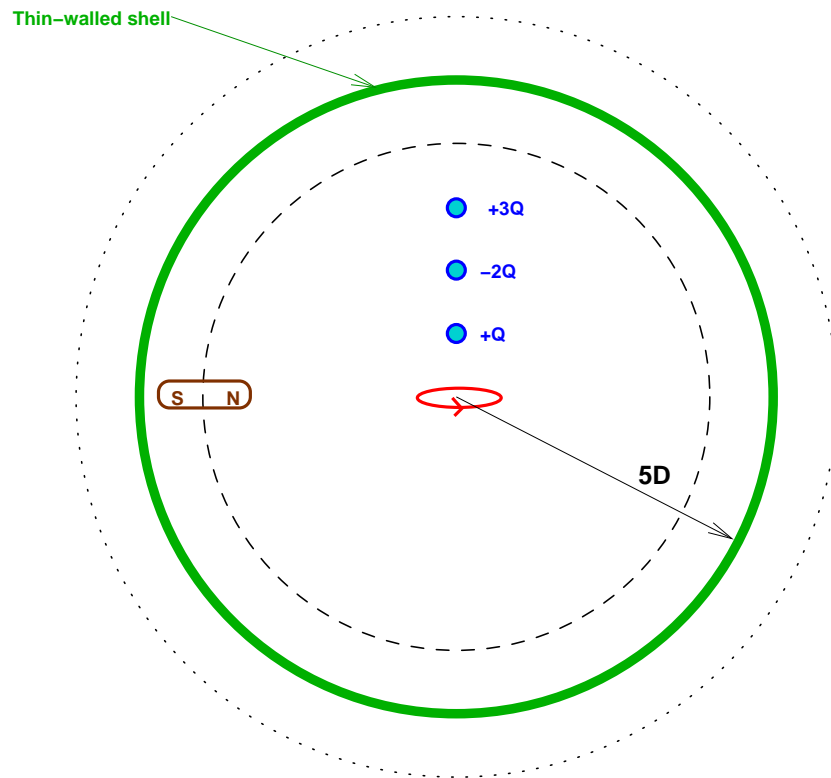


PHYS 122: – April 25, 2009
Slightly Revised Version April 29, 2009
Slightly Revised Again (fixed sign error in solution to PF3) May 2, 2009

More Practice Problems:

These are mostly from last year's final exam.

Be sure to try solving problems first without looking at solutions!

Problem PF1: Flux concept

A thin-walled electrically neutral conducting shell with given radius $R_{shell} = 5D$ encloses three point charges, one non-conducting permanent bar magnet, and one small current loop as shown above. The shell itself is *perfectly conducting*. The magnetic dipole moment associated with the current loop is given as μ_0 which is also precisely ten times greater than the dipole moment associated with the bar magnet. Assume that the above figure is drawn to scale so that the given point charges $+Q$, $-2Q$, and $+3Q$ are positioned at a given distances of D , $2D$, and $3D$ from the center of the sphere as shown. Assume that the current loop corresponds to a counter-clockwise rotation as seen from the above the sphere.

Part a) In terms of the parameters given, calculate the *total electric flux* through a concentric spherical surface that has a radius $\mathcal{R} = 4D$ (dashed circle). Explain your answer.

Part b) In terms of the parameters given, calculate the *total electric flux* through a concentric spherical surface that has a radius $\mathcal{R} = 6D$ (dotted circle). Explain your answer.

Part c) On the figure above, draw some *electric field lines* to indicate conceptually what the field in the region *outside* the conducting shell looks like. Is the field in this region spherically symmetric? Explain your work.

Part d) In terms of the parameters given, calculate the *total magnetic flux* through a concentric spherical surface that has a radius $\mathcal{R} = 4D$. Explain your answer.

Part e) Suppose we change the problem so that the perfectly conducting sphere shell is now replaced with a thin *perfectly insulating* spherical shell of the same size. How will this change your answers to parts (a), (b), and (c) above? Be specific. Explain your work.

Solution to Problem PF1

Part a) We use **Gauss' Law** to calculate the total flux:

$$\Phi_E \equiv \int_{\text{closed-surface}} \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$

So all we need to get the flux is to determine the charge enclosed. The surface at $\mathcal{R} = 4D$ shown with the dashed line encloses all three point charges. The other objects (such as the current loop and the bar magnet) are not charged. So the enclosed charge is just $Q_{\text{enc}} = +Q - 2Q + 3Q = 2Q$. Therefore the flux is given by:

$$\Phi_E = \frac{2Q}{\epsilon_0}$$

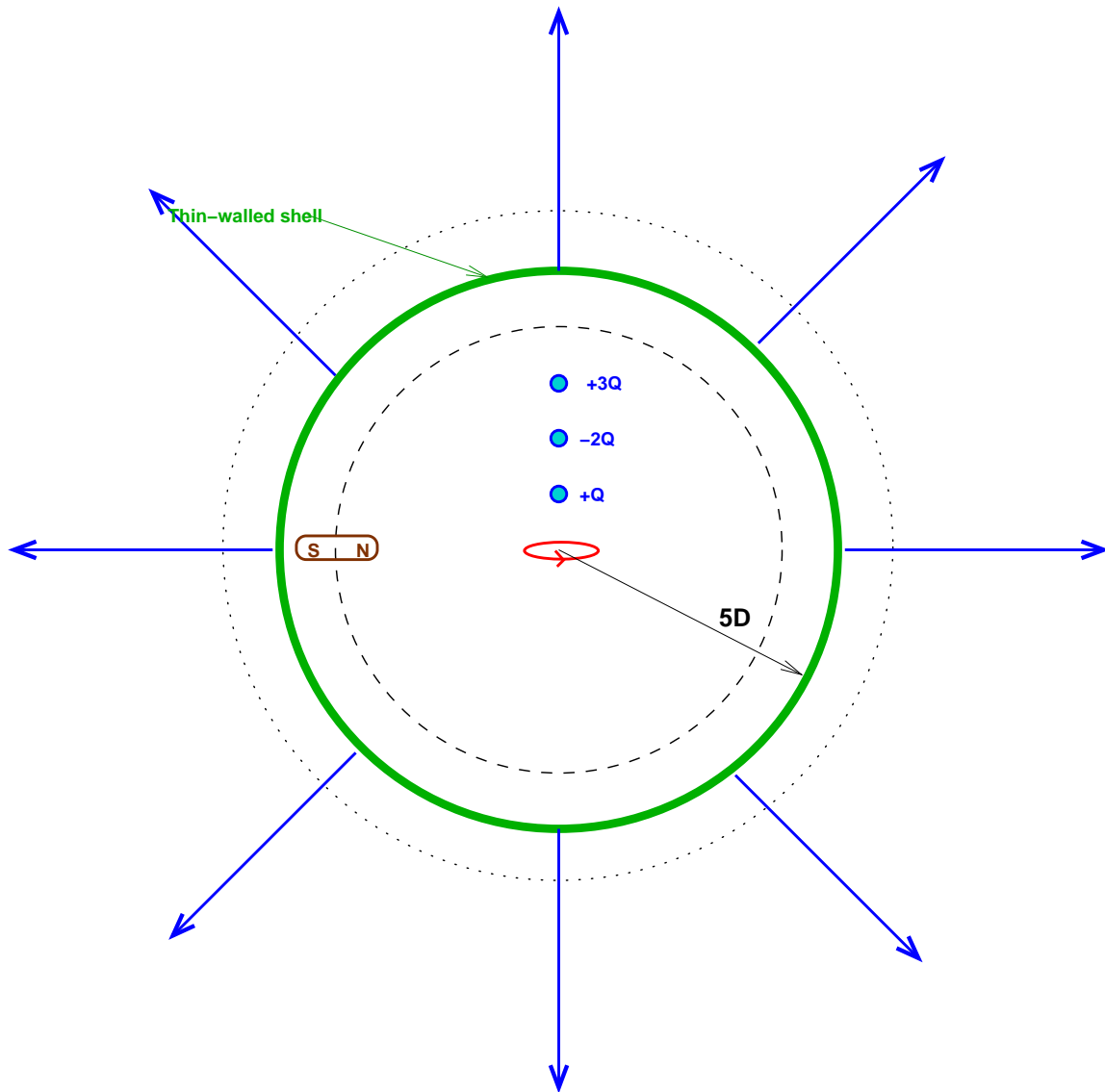
Part (b): For the radius at $\mathcal{R} = 6D$ we enclose the same charge, therefore the flux is the same as for Part (a):

$$\Phi_E = \frac{2Q}{\epsilon_0}$$

Note that since the shell is neutral, it does not change the enclosed charge. Note that fact that the shell is a perfect conductor does not mean that the electric field is blocked. Gauss' Law says you can use a conductor to shield from the influence of charges that are outside the conductor, not inside it.

Problem PF1 continues next page....

Part (c): In the region of the conductor material there is no electric field. This means that the asymmetric field due to the charged point particles inside the conductor is not “connected” to the charges on the outside of the conducting sphere. In other words the details of the specific configuration of charges inside the conductor is hidden, and the field outside the conductor is governed completely by the symmetry of the shape of the outside of the conductor. The field is spherically symmetric, radially outward. Field lines should be drawn to indicate uniform charge density on the outside surface:

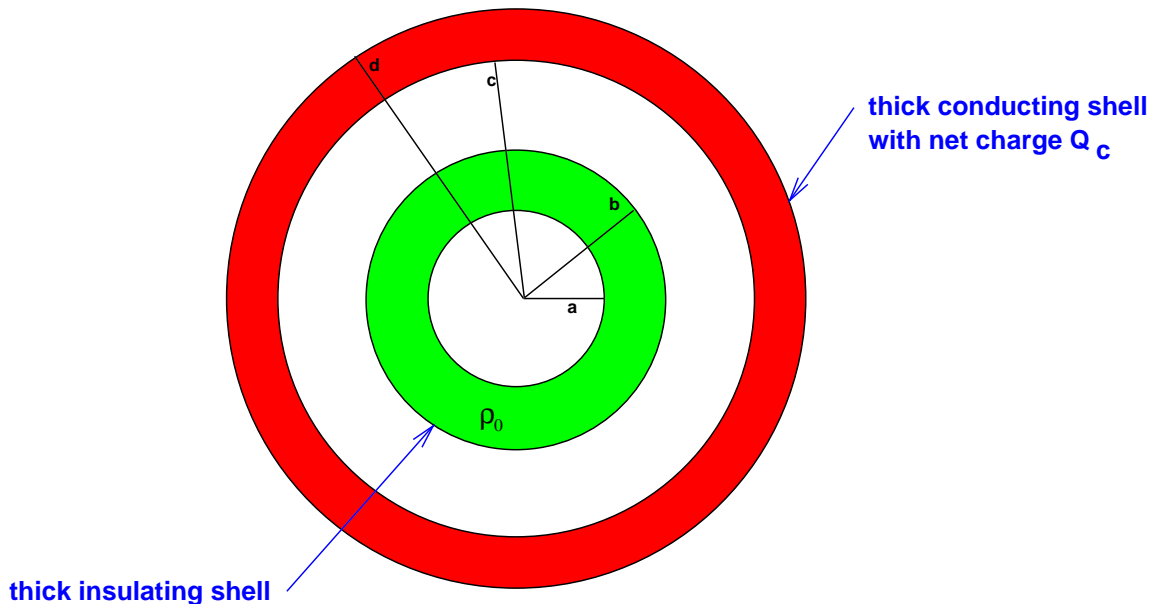


Problem PF1 continues next page....

Part (d): It doesn't matter at all what the radius is nor does it matter the configuration of loops or bar magnetic. The “**No-Name**” Law says that the total magnetic flux through any closed surface is **zero**. Period.

Part (e): The answers for Parts (a) and (b) above do not change in any way. Gauss' Law does not care if the surface is a conductor or an insulator: all that matters to the flux is the enclosed charge and that does not change.

However, the field for part (c) is no longer simple or spherically symmetric. A neutral insulator has no impact on electric field lines. So the the answer for Part (c) will change.

Problem PF2: Spherical Shells

Two thick concentric spherical shells are placed as shown above. The inner shell is made of *perfectly insulating material*. The outer shell is made of *perfectly conducting material*. The inner radius of the insulating shell is a , the outer radius of the insulating shell is b , the inner radius of the conducting shell is c , the outer radius of the conducting shell is d .

Suppose we embed charge on the inner shell with a given uniform charge density ρ_0 . Suppose also that we place a given net charge of Q_c on the conducting shell.

Part a) In terms of ρ_0 , a , and b , what is the total net charge Q_i of the insulating shell? Explain your work.

Part b) Use Gauss' Law to determine the electric field *everywhere*. Note that the four radii divide all space into five regions. Clearly indicate the value of the electric field for each of the five regions. Explain your work.

Part c) What is the *surface charge density* on the both the inner and outer surfaces of the conducting sphere? Explain your work. Hint: to earn full credit here, you should invoke Gauss' Law and your answer should completely avoid these three words: "attract", "force", and "induced".

Solution to Problem PF2

Part (a): Since the charge density is **uniform** as a function of position in the thick insulating shell, calculating the total charge Q_i is as simple as multiplying the density by the volume:

$$Q_i = (\text{charge density})(\text{volume of thick insulating shell})$$

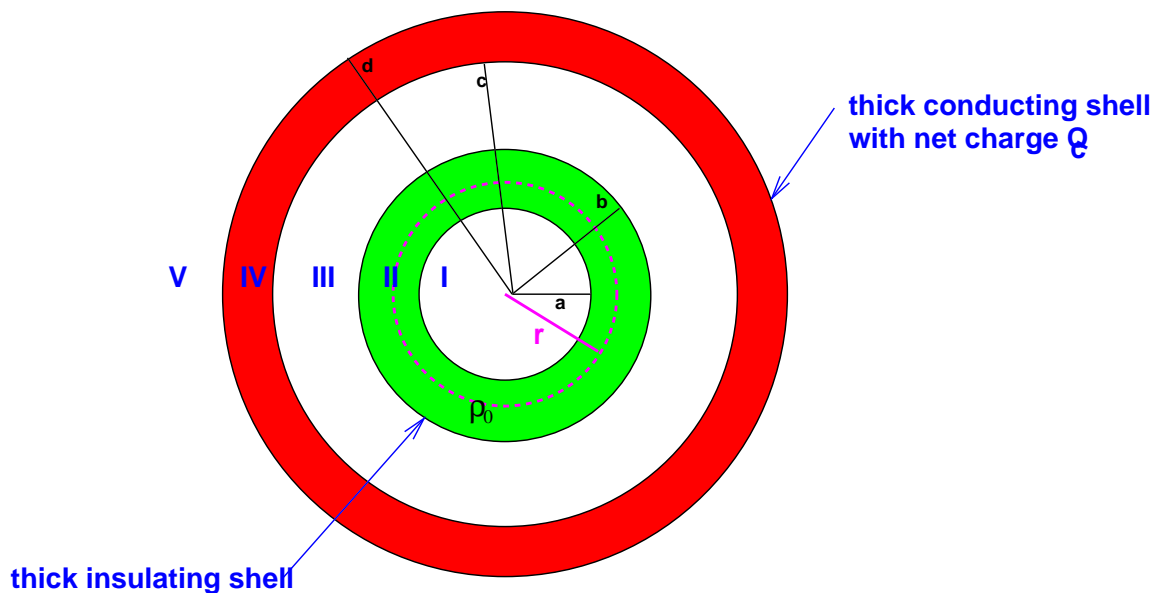
$$Q_i = (\text{charge density}) [(\text{volume of sphere of radius } b) - (\text{volume of sphere of radius } a)]$$

$$Q_i = \rho_0 \left[\left(\frac{4}{3} \pi b^3 \right) - \left(\frac{4}{3} \pi a^3 \right) \right]$$

$$Q_i = \frac{4\pi\rho_0(b^3 - a^3)}{3}$$

Part (b):

We need to calculate the electric field everywhere. For reference, we divide all space into **five** regions as a function of radius as show here:



We will use **Gauss' Law**:

$$\int_{\text{closed-surface}} \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$

We treat the left-hand side in the standard way: This side is the *flux* through a closed spherical *Gaussian surface* with appropriate symmetry at a given radius r . For example, the figure above shows such a surface for Region II shown as a dashed magenta sphere.

Since the charge distribution has spherical symmetry, we also expect the electric field to depend only on the radius.

$$\vec{E}(\vec{r}) = \vec{E}(r) = E(r)\hat{r}$$

In other words, at any given value of r , we can assume that the electric field has a particular magnitude and points outward from the center. This simplifies the surface integral calculation significantly:

$$\Phi_E = \int_{\text{surface}} [E(r)\hat{r}] \cdot d\vec{A}$$

Since \hat{r} and $d\vec{A}$ point parallel to each other at every point on the spherical Gaussian surface, the dot product gives us a scalar that is now a simply multiplication of the amplitudes:

$$\Phi_E = \int_{\text{surface}} E(r)dA$$

And we can pull out $E(r)$ since this is constant at a given r :

$$\Phi_E = E(r) \int_{\text{surface}} dA$$

And the integral of the differential area dA over a surface is just the surface area, which we know is $4\pi r^2$ for a sphere:

$$\Phi_E = E(r)(4\pi r^2)$$

So we put this into Gauss' Law and then we say for any spherical symmetry we can write:

$$E(r) = \left(\frac{1}{4\pi\epsilon_0} \right) \frac{Q_{\text{enclosed}}}{r^2}$$

Now, we calculate the enclosed charge region-by-region:

- **Region I:** For $r < a$ we see immediately that there is nothing enclosed – no charge, or anything else for that matter. So the field is given by:

$$\boxed{E_I(r) = 0}$$

- **Region II:** For $a < r < b$ we enclose a fraction of the thick insulating shell. The *Gaussian Surface* at radius r for $a < r < b$ encloses some, but not all of the charge Q_i . Since the charge density is *uniform* the amount of charge is just the charge density times the volume of the shell with inner radius a and outer radius r . This gives us an answer very similar to Part (a) substituting r for b :

$$Q_{II\text{enclosed}}(r) = \frac{4\pi\rho_0(b^3 - r^3)}{3}$$

Therefore:

$$E_{II}(r) = \left(\frac{1}{4\pi r^2} \right) \frac{\left(\frac{4\pi\rho_0(r^3 - a^3)}{3} \right)}{\epsilon_0}$$

$$\boxed{E_{II}(r) = \frac{\rho_0(r^3 - a^3)}{3r^2\epsilon_0}}$$

- **Region III:** For $b < r < c$ we are between the shells and so the enclosed charge is just the charge associated with the insulating shell:

$$Q_{III_{enclosed}}(r) = Q_{tot}$$

and so:

$$\boxed{E_{III}(r) = \frac{\rho_0(b^3 - a^3)}{3r^2\epsilon_0}}$$

- **Region IV:** For $c < r < d$ we are *embedded in a perfect conductor* and therefore we automatically know that the field is zero:

$$\boxed{E_{IV}(r) = 0}$$

- **Region V:** For $r > d$ the Gaussian surface encloses both the insulating and the conducting shells. So the enclosed charge is just the sum of these two charges:

$$Q_{V_{enclosed}}(r) = Q_i + Q_c$$

$$E_V(r) = \left(\frac{1}{4\pi\epsilon_0} \right) \frac{Q_i + Q_c}{r^2}$$

$$\boxed{E_V(r) = \left(\frac{1}{4\pi\epsilon_0} \right) \frac{\left(\frac{4\pi\rho_0(b^3 - a^3)}{3} \right) + Q_c}{r^2}}$$

Problem PF2 continues next page...

Part (c):

First we consider the charge induced on the *inner* surface of the conductor. We apply **Gauss' Law** for a Gaussian surface that is a sphere of radius r in **Region IV** corresponding to the conductor:

$$\int_{\text{closed surface}} \vec{E}_{IV} \cdot d\vec{A} = \frac{Q_{IV \text{ enclosed}}}{\epsilon_0}$$

We know that in the conductor, **the field is automatically zero**. So Gauss's Law tells us that the enclosed charge must also be zero. The enclosed charge includes the charge of the insulating shell Q_i plus any charge that appears on the inner surface of the conductor at radius c which we will call Q_{inner} . This has to total zero enclosed charge:

$$\begin{aligned} Q_i + Q_{inner} &= 0 \\ Q_{inner} &= -Q_i \\ Q_{inner} &= -\frac{4\pi\rho_0(b^3 - a^3)}{3} \end{aligned}$$

By symmetry, this charge should be uniformly spread across the surface. The surface charge density therefore is the total induced charge divided by the surface area of the inner surface of the conducting shell:

$$\begin{aligned} \sigma_{inner} &= \frac{Q_{inner}}{4\pi R_c^2} \\ \sigma_{inner} &= \frac{-\frac{4\pi\rho_0(b^3 - a^3)}{3}}{4\pi c^2} \end{aligned}$$

$$\boxed{\sigma_{inner} = -\frac{\rho_0(b^3 - a^3)}{3c^2}}$$

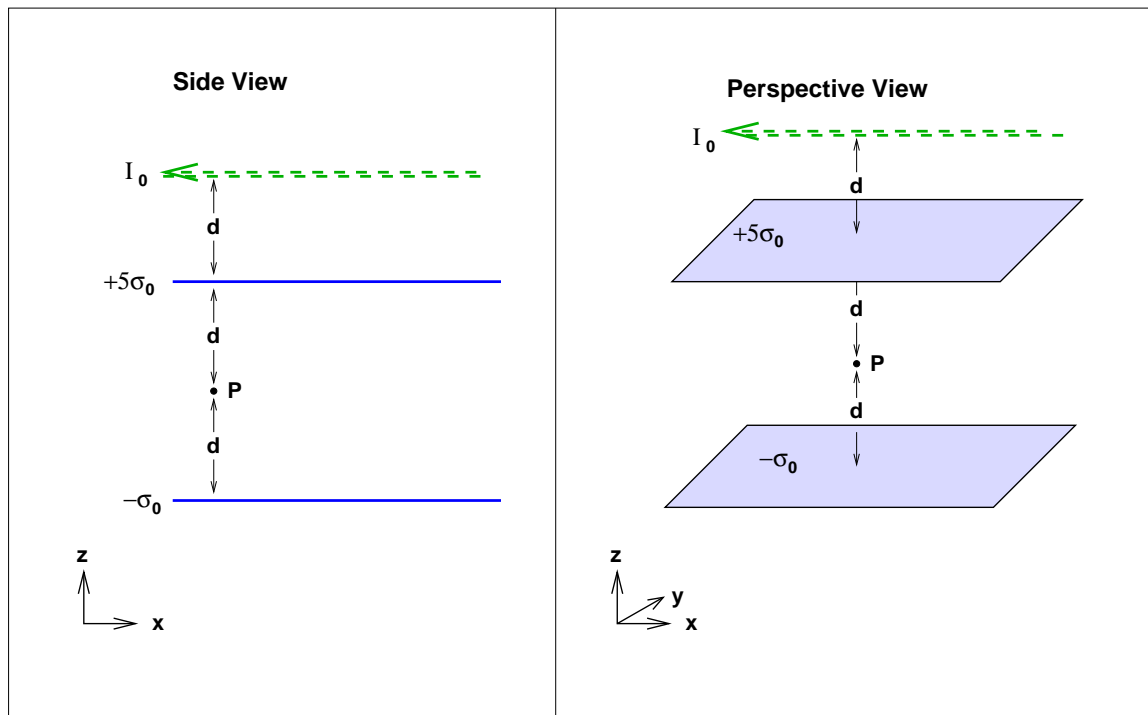
Problem PF2 Part (c) continues next page....

For the outer surface of the conductor, we know that the net charge of the conducting shell has to be Q_c . So the surface charge has the value that satisfies this constraint, given our answer for the charge induced on the inner surface:

$$\begin{aligned}Q_{outer} + Q_{inner} &= Q_c \\Q_{outer} &= Q_c - Q_{inner} \\Q_{outer} &= Q_c + \frac{4\pi\rho_0(b^3 - a^3)}{3}\end{aligned}$$

And again we divide this by the surface area to get the charge density:

$$\begin{aligned}\sigma_{outer} &= \frac{Q_{outer}}{4\pi R_d^2} \\ \sigma_{outer} &= \frac{Q_c + \frac{4\pi\rho_0(b^3 - a^3)}{3}}{4\pi d^2} \\ \sigma_{outer} &= \frac{Q_c}{4\pi d^2} + \frac{\rho_0(b^3 - a^3)}{3d^2}\end{aligned}$$

Problem PF3: Plates and Currents

The above plot shows an arrangement of two infinite charged plates. The upper plate has a given positive surface charge density $+5\sigma_0$ and the lower plate has a given negative surface charge density $-\sigma_0$. The separation between the plates is given as $2d$. Also, at a distance d above the upper plate is an infinite wire carrying a given current I_0 in the negative x -direction as shown.

Part a) Consider the point P which is located half-way between the plates. In terms of the given parameters, what is the *electric field* at point P ? Write down your answer as a *vector* using the coordinate system shown: x points right, y points back (into page) and z points up. Explain your work.

Part b) What is the *magnetic field* at point P ? Write down your answer as a *vector* using the coordinate system shown. Explain your work.

Part c) Now suppose a particle with a given mass m and a given negative charge $q = -q_0$ is placed at point P and is given an initial velocity $\vec{v} = v_0\hat{z}$. Is it possible to assign a positive value of the velocity v_0 that will allow the particle to move in a straight line without being deflected in any direction? *Here do not ignore gravity: Assume gravity results in a force on the particle downward, that is, in the $-\hat{k}$ direction.* If your answer is yes, then what is this value of v_0 in terms of the parameters given? If your answer is no, explain why not? Explain your work.

Solution to Problem PF3**Part (a):**

We know the electric field for an infinite uniform charged sheet with surface charge σ in the x-y plane:

$$\vec{E} = \pm \frac{\sigma}{2\epsilon_0} \hat{k}$$

where here the \pm symbol indicates that the field may be directed either in the positive or negative direction depending on whether we are above or below the sheet.

Determining the field due to two charged sheets is an exercise in **superposition**. Each charged plate contributes a **uniform electric field** at the Point P. The total field is the sum of the two fields together, one due to the bottom sheet and one due to the top sheet:

$$\begin{aligned}\vec{E}(P) &= \vec{E}_{bot}(P) + \vec{E}_{top}(P) \\ \vec{E}(P) &= \left(\frac{\sigma_{bot}}{2\epsilon_0} \hat{k} \right) + \left(-\frac{\sigma_{top}}{2\epsilon_0} \hat{k} \right)\end{aligned}$$

where here we note that since the upper sheet is above Point P the field is directed downward. Continuing:

$$\begin{aligned}\vec{E}(P) &= \left(\frac{-\sigma_0}{2\epsilon_0} \hat{k} \right) + \left(-\frac{+5\sigma_0}{2\epsilon_0} \hat{k} \right) \\ \vec{E}(P) &= -\left(\frac{6\sigma_0}{2\epsilon_0} \hat{k} \right) \\ \vec{E}(P) &= -\frac{3\sigma_0}{\epsilon_0} \hat{k}\end{aligned}$$

Note that both the upper sheet and the lower sheet result in a contribution to the field that is directed in the *downward* (negative k-hat) direction.

Note also that the answer to Part (a) does **not** depend on and distances (such as d) nor does the line of current have any impact on the electric field.

(Solution to Problem PF3 continues next page...)

Part (b):

For Part (b) we note that the only thing that creates or impacts the magnetic field is the line of current. Static charged plates, whether conducting or not, have no impact on the magnetic field. We know the expression for the magnetic field and the only tricky thing is working out the **tangential direction** in accordance with the curly-fingered **Right-Hand-Rule**:

$$\vec{B}(r) = \left(\frac{\mu_0 I_{wire}}{2\pi r} \right) \hat{\mathcal{T}}$$

where $\hat{\mathcal{T}}$ corresponds to the tangential vector. At Point P the distance from the wire $r = 2d$ as can be seen from the figure:

$$\vec{B}(P) = \left(\frac{\mu_0 I_0}{2\pi(2d)} \right) \hat{\mathcal{T}}$$

And finally we use the Right-Hand-Rule to convince ourselves that the tangential vector field points **out of the page** corresponding to the **negative-y** direction:

$$\boxed{\vec{B}(P) = - \left(\frac{\mu_0 I_0}{4\pi d} \right) \hat{j}}$$

(Solution to Problem PF3 continues next page...)

Part (c):

First we consider the implications of a particle moving with *constant velocity*. According to **Newton's Second Law** such a particle must experience **zero net force**. In this problem there are three possible forces on the particle: (1) Electric, (2) Magnetic, and (3) Weight (due to gravity). So for the particle to move in a straight line it must be true that:

$$\vec{F}_{net} = \vec{F}_{elec} + \vec{F}_{mag} + \vec{F}_{weight} = 0$$

Let's consider these forces in turn, using the fact that we already calculated the field in Parts (a) and (b): We know that the **Force Rule for a point charge in an electric field** is given by:

$$\begin{aligned}\vec{F}_{elec} &= q_0 \vec{E}(P) \\ \vec{F}_{elec} &= (-q_0) \left(-\frac{3\sigma_0}{\epsilon_0} \hat{\mathbf{k}} \right) \\ \vec{F}_{elec} &= \frac{3q_0\sigma_0}{\epsilon_0} \hat{\mathbf{k}}\end{aligned}$$

Likewise, we know the force rule for a point charge moving through a magnetic field which is given by:

$$\begin{aligned}\vec{F}_{mag} &= q\vec{v} \times \vec{B}(P) \\ \vec{F}_{mag} &= (-q_0)(v_0\hat{\mathbf{i}}) \times \left(-\frac{\mu_0 I_0}{4\pi d} \right) \hat{\mathbf{j}} \\ \vec{F}_{mag} &= \left(\frac{q_0 v_0 \mu_0 I_0}{4\pi d} \right) (\hat{\mathbf{i}} \times \hat{\mathbf{j}}) \\ \vec{F}_{mag} &= \left(\frac{q_0 v_0 \mu_0 I_0}{4\pi d} \right) \hat{\mathbf{k}}\end{aligned}$$

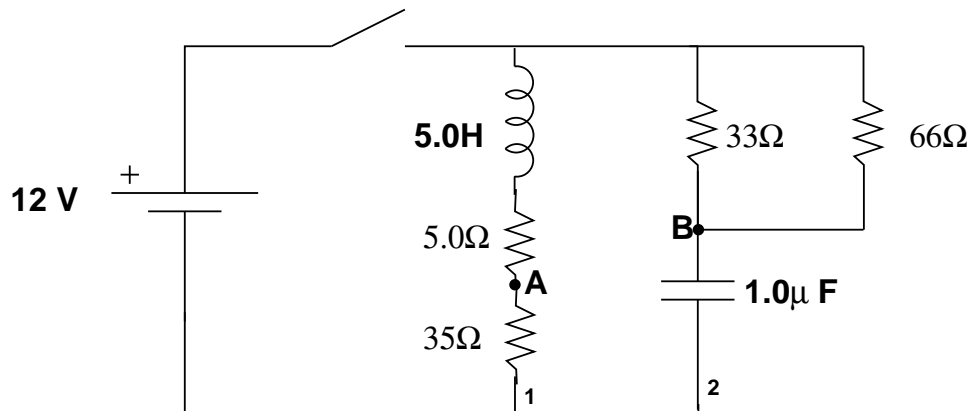
And finally for gravity we use the old result from mechanics:

$$\vec{F}_{weight} = -m_0 g \hat{\mathbf{k}}$$

Since every force is directed in the vertical ($\hat{\mathbf{k}}$) direction, we can just add the scalar components with the appropriate sign, set this total equal to zero and solve for v_0 :

$$\begin{aligned}F_{elec_z} + F_{mag_z} + F_{weight_z} &= 0 \\ \left(\frac{3q_0\sigma_0}{\epsilon_0} \right) + \left(\frac{q_0 v_0 \mu_0 I_0}{4\pi d} \right) - m_0 g &= 0 \\ \left(\frac{q_0 v_0 \mu_0 I_0}{4\pi d} \right) &= m_0 g - \left(\frac{3q_0\sigma_0}{\epsilon_0} \right) \\ v_0 &= \left(\frac{4\pi d}{q_0 \mu_0 I_0} \right) \left(m_0 g - \frac{3q_0\sigma_0}{\epsilon_0} \right)\end{aligned}$$

Perhaps this can be simplified a little further, but why bother....

Problem PF4:

A circuit is assembled as shown above. The inductor has a value $L = 5.0$ henrys. The capacitor has a value of 1.0 micro-farads. The values of five resistors are shown above. The circuit is attached to a sturdy 12 volt battery as shown and then the switch is closed.

Part a) What is the current through the inductor *immediately* after the switch is closed? Explain how you know this.

Part b) What is the approximate voltage measured at point B relative to point A at a time 10 seconds after the switch is closed? Give your answer in volts. Explain.

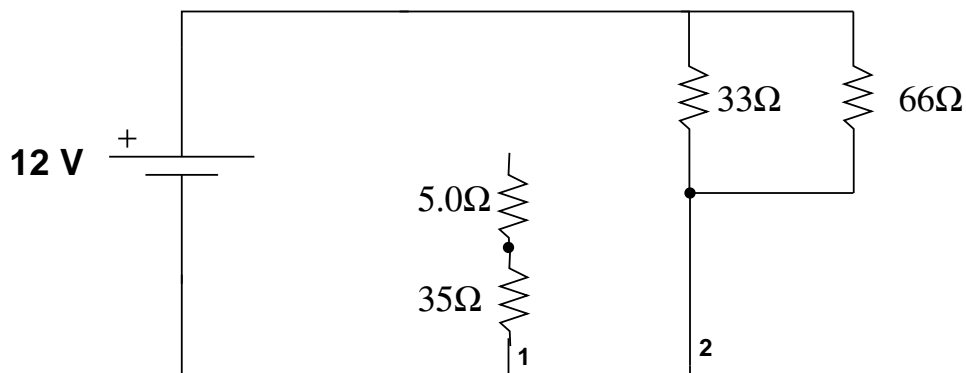
Part c) What is the approximate current through the voltage source when measured at a time 0.10 seconds after the switch is closed? Give your answer in amps. Explain.

Solution to Problem PF4

Note: I think this is the only exam problem I gave all last year where a calculator would be helpful. Perhaps such a problem will appear on this year's final exam as well.

Part (a):

This is a circuit with RL and RC arms, so we are generally expecting some kind of exponential time dependence when the switch is thrown. When we hear the word “*immediately*” after the switch is closed, we think “short-time solution” regardless of the value of the time constants. For an inductor, this corresponds to an open switch (a gap). For a capacitor this corresponds to a closed switch (a wire). Conceptually the re-drawn circuit looks like this:



Current cannot flow through Arm 1, so the current through the inductor immediately after the switch is thrown is **Zero amps**.

(Solution to Problem PF4 continues next page...)

Part (b):

To consider what is happening at a time $t = 10$ seconds we need to ask this very important question: **How does t compare with the time constants associated with each arm?**

Note in particular: *Simply stating that “10 seconds is a long time” without actually calculating the time constants for each arm is just wrong!* If you skip this step you are committing a grave logical error. Any given time is only “long” compared to some other time. The time between now and lunch is either long or short compared to other times. The span of all human history is very very short compared to the age of even a comparatively young star like our Sun.

For Arm 1, the time constant is for an “RL” circuit:

$$\tau_1 = \frac{L}{R_1}$$

Here we have need to use the equivalent resistance of the two resistors in series in Arm 1. We add resistors in series:

$$R_1 = 5 \Omega + 35 \Omega = 40 \Omega$$

Therefore, plugging in numbers:

$$\tau_1 = \frac{5 \text{ Henrys}}{40 \Omega} = \frac{1}{8} \text{ seconds}$$

Therefore $t \gg \tau_1$ and so we are in the long-time regime for Arm 1.

For the capacitor we have a similar calculation:

$$\tau_2 = R_2 C$$

Here we have need to use the equivalent resistance of the two resistors in parallel in Arm 2. We use the rule for resistors in parallel:

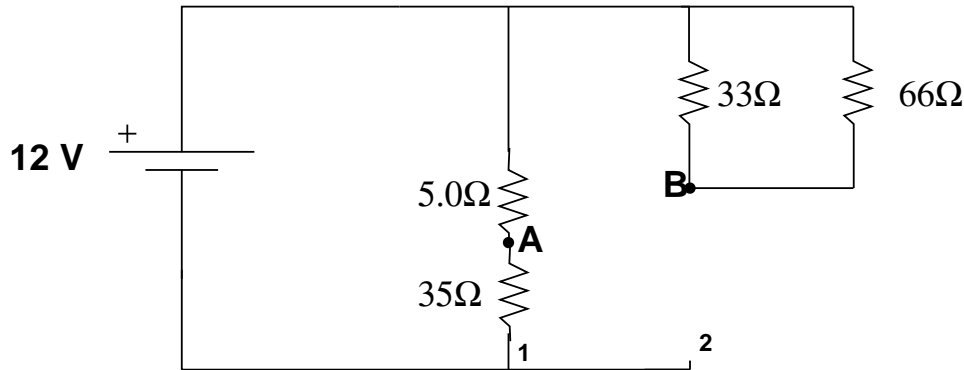
$$\begin{aligned} \frac{1}{R_2} &= \frac{1}{33 \Omega} + \frac{1}{66 \Omega} \\ \frac{1}{R_2} &= \frac{2}{66 \Omega} + \frac{1}{66 \Omega} \\ \frac{1}{R_2} &= \frac{3}{66 \Omega} \\ \frac{1}{R_2} &= \frac{1}{22 \Omega} \\ R_2 &= 22 \Omega \end{aligned}$$

Therefore, plugging in numbers:

$$\tau_2 = (22 \Omega)(1 \times 10^{-6} \text{ Farads}) = 2.2 \times 10^{-5} \text{ seconds}$$

Therefore $t \gg \tau_2$ and so we are in the long-time regime for Arm 2.

Since we can use the long-time approximates for both the inductor (closed switch) and the capacitor (open switch) we can re-draw the circuit like this:



So finally, we can get back to the question of the voltage at Point B relative to Point A. In order to figure this out, we need to consider how the voltage is determined at each of these two points:

First, we note that since the current through Arm 2 is zero, there can be no voltage drop across either the 33 or the 66 Ohm resistors, which means that the voltage at **Point B** is the same as the voltage at the top of Arm 2 which is also the same as the voltage at the top of Arm 1, which is **also** the voltage at the top of the battery. This corresponds to a voltage of 12 volts relative to the bottom of the battery, for example.

To get the voltage at **Point A** we notice that the structure with two resistors in series like this corresponds to a “**Voltage Divider**”. If you have forgotten how a Voltage Divider works, you can solve from first principles using Ohm’s Law to calculate the current, and then working out the voltage drop across the top resistor:

$$V_1 = I_1 R_1$$

$$I_1 = \frac{V_1}{R_1}$$

$$I_1 = \frac{12 \text{ Volts}}{40\Omega}$$

$$I_1 = 0.3 \text{ Amps}$$

Then applying Ohm’s Law to the top resistor:

$$V_{5\Omega} = I_{5\Omega} / R_{5\Omega}$$

$$V_{5\Omega} = (0.3 \text{ Amps})(5 \Omega)$$

$$V_{5\Omega} = 1.5 \text{ Volts}$$

Alternatively, we can get this result by applying directly the standard relationship for a Voltage Divider:

$$V_{5\Omega} = V_{bat} \left(\frac{R_{5\Omega}}{R_{5\Omega} + R_{35\Omega}} \right)$$
$$V_{5\Omega} = (12 \text{ Volts}) \left(\frac{5\Omega}{5\Omega + 35\Omega} \right)$$
$$V_{5\Omega} = (12 \text{ Volts}) \left(\frac{1}{8} \right)$$
$$V_{5\Omega} = 1.5 \text{ Volts}$$

And since the voltage drop from the top of Arm 1 to Point A is the same thing as the voltage at Point B with respect to Point A we can say:

$$\boxed{V_{\text{B-relative-to-A}} = V_{5\Omega} = 1.5 \text{ Volts}}$$

(Solution to Problem PF4 continues next page...)

Part (c):

Now we are asked for the current through the battery at time $t = 0.1$ seconds. This time is certainly long compared to $\tau_2 = 2.2 \times 10^{-5}$ seconds, so we can treat the capacitor as an open switch. However, this time is **comparable** to the time $\tau_1 = \frac{1}{8}$ seconds. So we can use *neither* the short-time solution, *nor* the long-time solution for the inductor. Instead we must use the proper **exponential form**: We have two general forms to consider using for the current:

$$X(t) = X_0 \exp\left(-\frac{t}{\tau}\right)$$

OR

$$X(t) = X_f \left[1 - \exp\left(-\frac{t}{\tau}\right)\right]$$

In this case, we consider that in the short-time limit we start with zero current through the inductor and in the long-time limit we expect a steady-state current to flow through the inductor. So the *second form* of the two equations above is the correct one to use:

$$I_1(t) = I_f \left[1 - \exp\left(-\frac{t}{\tau_1}\right)\right]$$

The only term we have not worked out is I_f corresponding to the steady state current through Arm 1. We already calculated this for Part (b) using Ohm's Law:

$$I_f = I_1(\text{steady-state}) = \frac{V_1}{R_1}$$

$$I_f = \frac{12 \text{ Volts}}{40\Omega}$$

$$I_f = 0.3 \text{ Amps}$$

Note also that since there is no current through Arm 2, the battery is in series with Arm 2 and therefore the current through the battery is in equal to the current through Arm 2:

$$I_{bat}(t = 0.1 \text{ sec}) = I_1(t = 0.1 \text{ sec})$$

$$= I_f \left[1 - \exp\left(-\frac{t}{\tau_1}\right)\right]$$

Plugging in numbers:

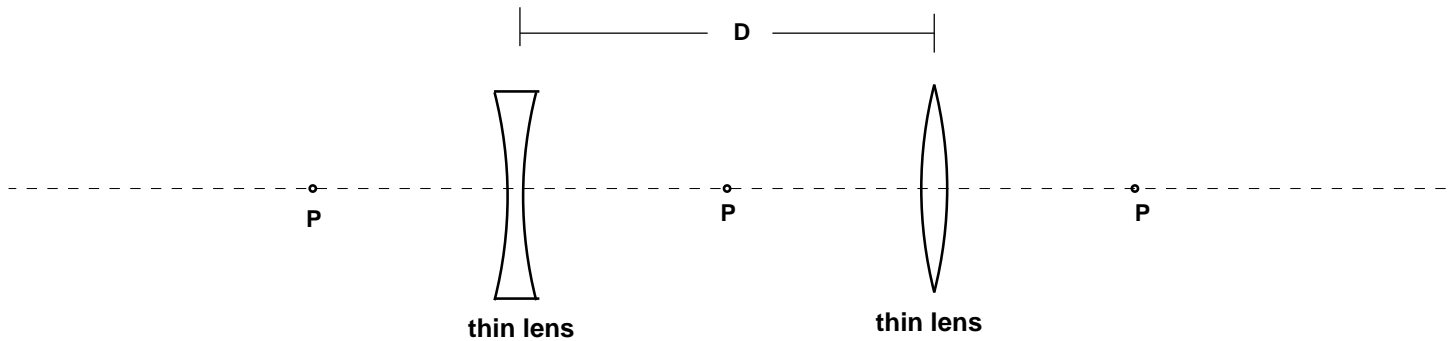
$$= (0.3 \text{ Amps}) \left[1 - \exp\left(-\frac{0.1 \text{ seconds}}{\frac{1}{8} \text{ seconds}}\right)\right]$$

$$= (0.3 \text{ Amps}) [1 - \exp(-0.8)]$$

$$= (0.3 \text{ Amps})(1 - 0.449)$$

$$= (0.3 \text{ Amps})(0.551)$$

$$\boxed{I_{bat} = 0.165 \text{ Amps}}$$

Problem PF5: Optics

Two thin lenses are positioned as shown above. The distance between the two lenses is given as D . The left lens is *concave* and the right lens is *convex*. Each lens has the same given focal length f . You also know that D is precisely twice the focal length f . Primary and secondary focal points are drawn to scale above.

In this problem the *left* lens creates an image of the *right* lens, and the *right* lens creates an image of the *left* lens.

Part (a) – Determine the position of each of the two images using the lens equation.

Part (b) – Verify your result to part (a) by using ray tracing for at least two principle rays for each image.

Part (c) – Also calculate the magnification for each image, and for each of the two images, determine if the image is *real* or *virtual* and whether or not the image is *inverted*. Explain your work here.

Solution to Problem PF5

Part (a): The lens equation says:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$

where s is the coordinate of the object, s' is the coordinate of the image and f is the focal length.

Let start by considering the image of the **concave** lens that is made by the **convex** lens. In other words the 'object' is the lens on the left. We see from the figure that $s = D$. We write down the lens equation and solve for s' :

$$\begin{aligned}\frac{1}{D} + \frac{1}{s'} &= \frac{1}{f} \\ \frac{1}{D} + \frac{1}{s'} &= \frac{1}{D/2}\end{aligned}$$

Algebra:

$$\begin{aligned}\frac{1}{s'} &= \frac{2}{D} - \frac{1}{D} \\ \frac{1}{s'} &= \frac{1}{D} \\ \boxed{s' = D = 2f}\end{aligned}$$

The coordinate of the image is *positive* corresponding to the far side of the convex lens (outgoing) which in this case corresponds to a position to the **right** of the convex lens.

Next, to calculate the position of the image of the **convex** lens formed by the **concave** lens we note again that $s = D$ but now since we have a diverging lens, the focal length is negative:

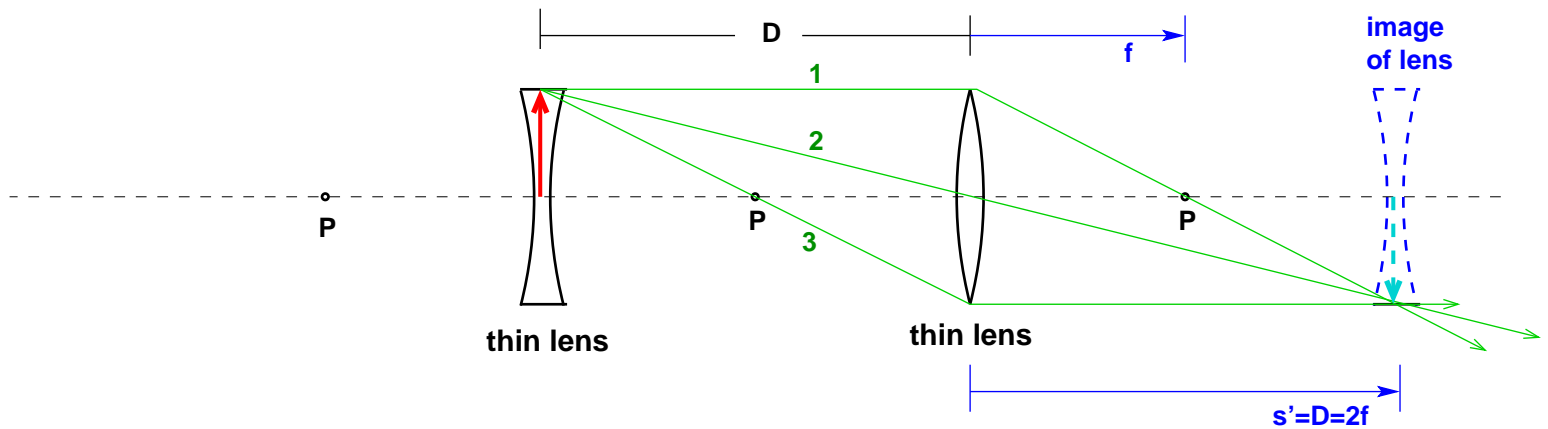
$$\begin{aligned}\frac{1}{D} + \frac{1}{s'} &= \frac{1}{f} \\ \frac{1}{D} + \frac{1}{s'} &= -\frac{1}{D/2} \\ \frac{1}{s'} &= -\frac{2}{D} - \frac{1}{D} \\ \frac{1}{s'} &= -\frac{3}{D} \\ \boxed{s' = -\frac{D}{3} = -\frac{2f}{3}}\end{aligned}$$

Since s' is negative, so we expect to find the image on the we expect to find the image on the *near* side (incoming) of the concave lens corresponding to a position to the **right** of the concave lens.

(Solution to Problem PF5 continues next page...)

Part (b):

We can confirm our results from Part (a) with ray tracing. Here's the first case where the object is the **concave** lens and the **convex** lens is creating the image:



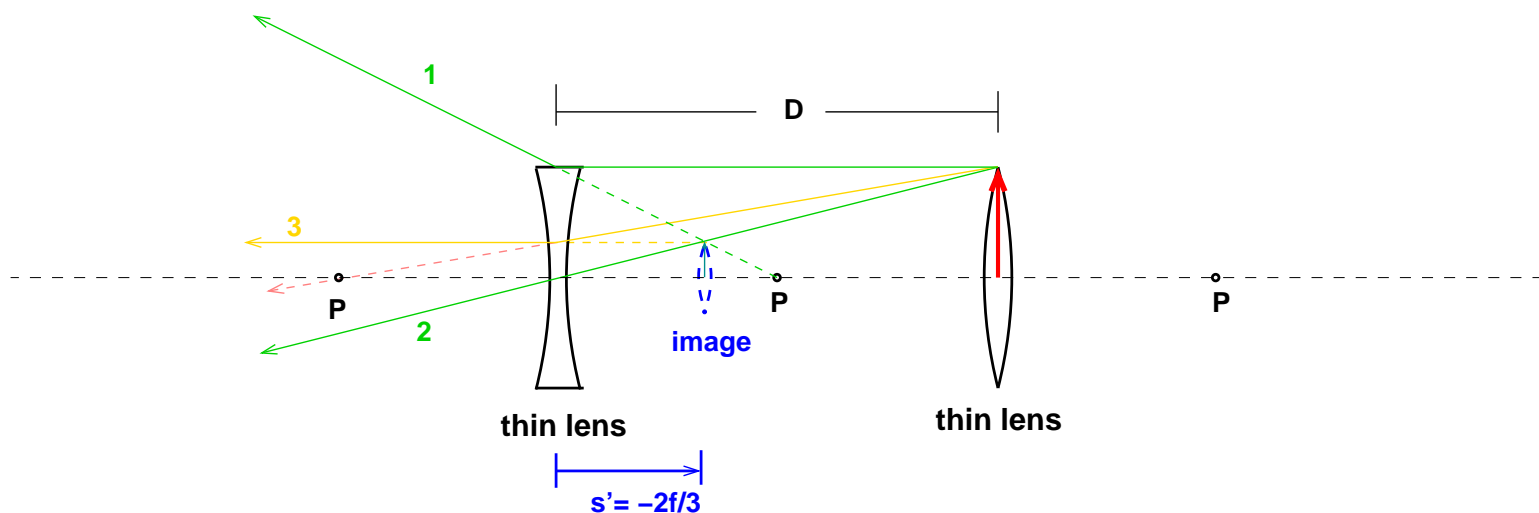
To emphasize the fact that the concave lens is the **object** we have drawn a red arrow on the top side. This will help us later with understanding precisely where the image is, also. Note that to create an image we pick the “top point” of the object (concave lens) and draw three principle rays drawn above as indicated:

1. A ray that travels from the object in the direction of the axis of the system which is then refracted at the convex lens toward the *far* focal point P ,
2. A ray that travels from the object straight toward the center of the convex lens (this ray is not refracted).
3. A ray that travels toward the *near* focal point, emerging from the convex lens in a line parallel to the axis of the system.

All of these rays intersect at one point, allowing up to locate the *image* as shown (blue dashed) at a distance $D = 2f$ from the convex lens, as expected.

(Solution to Problem PF5 part (b) continues next page...)

Here's the second case where the object is the **convex** lens and the **concave** lens is creating the image:



Again, draw a red arrow to emphasize the fact that now the **convex** lens is the object. we pick the “top point” of the object (convex lens) and draw three principle rays drawn above as indicated. Solid lines correspond to real light rays, while dashed lines correspond to extension of these rays through the optics. Note that the third ray is quite tricky and so is described with yellow and pink lines as discussed below to avoid confusion with the other (green) rays.

1. A ray that travels from the object in the direction of the axis of the system which is then refracted at the concave lens to a path that points *away from the near focal point* P . The solid green line here represents the actual path of the light ray. The dashed green line represents the *apparent* path of the ray from the (virtual) image.
2. A ray that travels from the object straight toward the center of the concave lens (this ray is not refracted). For this one ray, both the real and apparent directions of the ray are the same at every point.
3. A ray that travels toward the *far* focal point, emerging from the concave lens in a line parallel to the axis of the system (This rays follows the reverse path of a parallel ray into the other side of the lens). The dashed pink line represents the path that the ray would have taken if it had not been refracted by the lens. The solid yellow line represents the actual path of the ray. The dashed yellow line represents the *apparent* path of the ray from the (virtual) image.

All of these rays (as seen from the far side of the concave lens) *apparently* intersect at one point, allowing up to locate the *image* as shown (blue) at a distance $s' = -\frac{D}{3} = -\frac{2f}{3}$ from the concave lens, as expected.

(Solution to Problem PF5 continues next page...)

Part (c):

In general the magnification is given by the ratio:

$$M = \frac{s'}{s}$$

For the first case where the object is the concave lens and the image is created by the convex lens, we get:

$$M = \frac{s'}{s} = \frac{D}{D} = 1$$

This matches what we see in the ray trace. We also note that this image is a **real and inverted** image because the rays are really present at the image and because rays from the top of the object end up at the bottom of the image.

For the second case where the object is the convex lens and the image is created by the concave lens, we get:

$$M = \frac{s'}{s} = \frac{D/3}{D} = \frac{1}{3}$$

This matches what we see in the ray trace. We also note that this image is a **virtual and not inverted** image because the rays only *appear to be originating from the image* and because rays from the top of the object end up (appearing to arrive from) the top of the image.

Problem PF6: Maxwell's Revenge

Suppose an electric potential function is defined within some arbitrary region as a position of both time and space in accordance with the following parametrization:

$$V(x, y, z, t) = Axyz^2e^{-\left(\frac{t}{\tau}\right)} + Bzt^{-\alpha} + C \cos(\omega t)$$

Here, A , B , C , α , τ and ω are given positive constants with the appropriate units.

Part (a) Determine the Electric Field as a function of position and time. Explain how you got your answer. Hint: Don't forget, the electric field is a *vector*.

Part (b) Suppose we define a *loop* which is a flat square that bounds a unit area on the x-y plane. The four corners of the square in Cartesian coordinates (x, y, z) are therefore $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, and $(0, 1, 0)$. In terms of the given parameters, what is the *path integral* of the magnetic field integrated around this loop? Explain your answer.

Solution to Problem PF6:

We start with the definition of the electric potential:

$$V(x, y, z, t) = Axyz^2 e^{-\left(\frac{t}{\tau}\right)} + Bzt^{-\alpha} + C \cos(\omega t)$$

To find the electric field from this potential, we calculate the **gradient**:

$$\vec{\nabla} \equiv \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

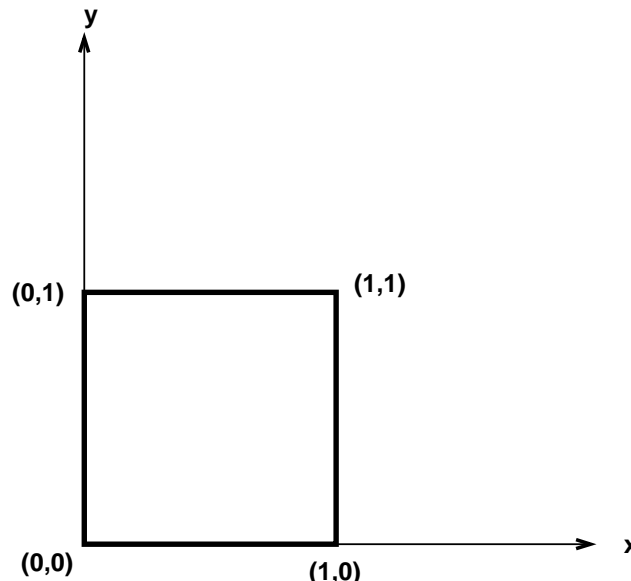
Note that in all of this, we treat the time t as a arbitrary parameter that does *not* depends on position.

We note that the first term includes x , y , and z . The second term only depends on z and the third term does not depend on any of these. Applying the derivative operator:

$$\vec{E} = -\vec{\nabla}V = \hat{i} \left[-Ayz^2 e^{-\left(\frac{t}{\tau}\right)} \right] + \hat{j} \left[-Axz^2 e^{-\left(\frac{t}{\tau}\right)} \right] + \hat{k} \left[-2Axyz e^{-\left(\frac{t}{\tau}\right)} - Bt^{-\alpha} \right]$$

Note in particular that the term in the potential with the coefficient C has no impact on the electric field because this term is “constant with position”.

Now we consider the path integral around a loop in the x-y plane (z equal zero) as shown:



We are asked to calculate the magnetic field around this loop. At first blush, this seems impossible as we are given no specification for the magnetic field. But we can take advantage of **Maxwell’s Extension to Ampere’s Law**: as follows:

$$\int_{\text{closedloop}} \vec{B} \cdot d\vec{\ell} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

So we can *change* the problem of calculating a *path integral* into a new problem of calculating a *surface flux*. Since the surface is flat on the x-y plane, then only the component of the field aligned with the \hat{k} vector contributes to the flux. Furthermore, since we are in the x-y plane, $z = 0$ and the first term that contributes to the \hat{k} component goes to zero. In fact, we see that the flux through the square is spatially uniform:

$$\begin{aligned}\Phi_E &= \int_{area} \vec{E} \cdot d\vec{A} \\ \Phi_E &= \int_{area} -Bt^{-\alpha} dA \\ \Phi_E &= -Bt^{-\alpha} \int_{area} dA\end{aligned}$$

The area of the square is one unit:

$$\Phi_E = -Bt^{-\alpha}$$

The time derivative is easy:

$$\frac{d\Phi_E}{dt} = \alpha Bt^{-(\alpha+1)}$$

And so the path integral then is:

$$\boxed{\int_{closedloop} \vec{B} \cdot d\vec{\ell} = \mu_0 \epsilon_0 \alpha Bt^{-(\alpha+1)}}$$

Caveat: The above calculation assumes that the current density function $\vec{J}(\vec{r}, t)$ is zero everywhere so that there is no net current going through the loop (which according to Ampere's Law will also contribute to the path integral). In fact, such an assumption is may be inconsistent with with our rather unusual form of the potential function.