Part (a)

We are given the charge distribution function as:

$$\rho(z) = \rho_0 \left| \frac{z}{K} \right|$$

where ρ_0 represents a positive constant volume charge density (charge per unit volume). In other words, rho_0 is a constant that already has units of charge density, which is what we want for $\rho(z)$. This means that the ratio $\frac{z}{K}$ must be *unit-less* and therefore the constant K must be the same units as the cartesian coordinate z. In other words, $\boxed{K \text{ must be a } \textbf{length}}$.

Part (b)

Our approach will be as follows:

- We will use Gauss' Law together with the symmetry of the charge density to determine the electric field everywhere as a function of z.
- We will use the definition of the electric potential to determine the voltage given the electric field.

First, let's get the field from the charge using Gauss' Law. We define $z = 0$ as the axis of symmetry and apply a **Gaussian pillbox** as our closed Gaussian Surface:

$$\int_{pillbox} \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{\epsilon_0}$$

We know from the symmetry that the flux on the left hand side is determined by the area of the endcaps times the field at any given position (since the sidewalls contribute zero to the flux):

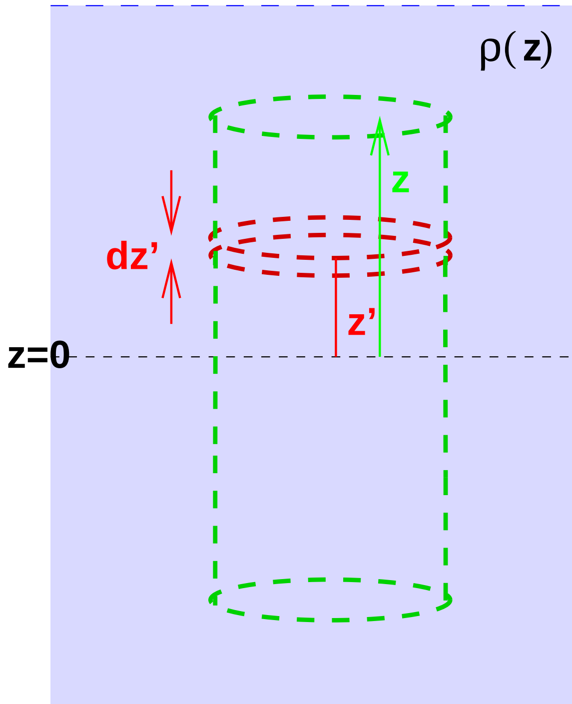
$$\int_{pillbox} \vec{E} \cdot d\vec{A} = 2AE(z)$$

Therefore:

$$E(z) = \frac{Q_{enclosed}}{2A\epsilon_0}$$

We note here that since we have planar symmetry, the field in the negative region is the same as the positive one only in the opposite direction.

Now we need to calculate the total enclosed charge. We do this using the standard method of integral calculus applies to Pillbox:



We to get the total enclosed charge we integrate the differential charge element associated with a small "coin" of charge of area A and thickness dz' as follows:

$$Q_{enclosed} = \int dQ$$

$$Q_{enclosed} = \int_{z'=-z}^{z'=z} \rho(z')\, dV'$$

where dV' is the differential volume of the little coin, corresponding to the area times the thickness: $dV' = A\, dz'$.

$$Q_{enclosed} = \int_{z'=-z}^{z'=z} \rho(z') A\, dz'$$

We also take advantage of the symmetry to argue that the total charge enclosed is twice the charge enclosed in the positive region only:

$$Q_{enclosed} = 2\int_{z'=0}^{z'=z} \rho(z') A\, dz'$$

$$Q_{enclosed} = 2\int_{z'=0}^{z'=z} \rho_0 \left(\frac{z'}{K}\right) A\, dz'$$

where we ignore the "absolute value" bars because we are now working only in the $z > 0$ region:

$$Q_{enclosed} = \frac{2\rho_0 A}{K}\int_{z'=0}^{z'=z} z'\, dz'$$

$$Q_{enclosed} = \frac{2\rho_0 A}{K}\left[\frac{z'^2}{2}\right]_{z'=0}^{z'=z}$$

$$Q_{enclosed} = \frac{2\rho_0 A}{K}\left(\frac{z^2}{2}\right)$$

$$Q_{enclosed} = \frac{\rho_0 A z^2}{K}$$

Therefore:

$$E(z) = \frac{\left(\frac{\rho_0 A z^2}{K}\right)}{2A\epsilon_0}$$

$$E(z) = \frac{\rho_0 z^2}{2K\epsilon_0} \tag{1}$$

Now we need to get the voltage. To do this we need to *choose* an appropriate zero-point reference corresponding to zero volts. In this problem, the natural place to do this is to assign $V_{ref} = 0$ at $z_{ref} = 0$. We write down the voltage equation then:

$$V(z) - V_{ref}(z_{ref}) = -\int_{z'=z_{ref}}^{z'=z} E(z')dz'$$

$$V(z) - 0 = -\int_{z'=0}^{z'=z} E(z')dz'$$

Here we have already taken advantage of the 1-D aspect of the problem.

$$V(z) = -\int_{z'=0}^{z'=z} \frac{\rho_0 z^2}{2K\epsilon_0}dz'$$

$$V(z) = -\frac{\rho_0}{2K\epsilon_0}\int_{z'=0}^{z'=z} z'^2 dz'$$

$$V(z) = -\frac{\rho_0}{2K\epsilon_0}\left[\frac{z'^3}{3}\right]_{z'=0}^{z'=z}$$

$$V(z) = -\frac{\rho_0}{2K\epsilon_0}\left(\frac{z^3}{3}\right)$$

$$\boxed{V(z) = -\frac{\rho_0 z^3}{6K\epsilon_0}}$$

Part (c)

To get the magnitude and direction of the electric field, we apply the **gradient operator**:

$$\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r})$$

where

$$\nabla \equiv \hat{\boldsymbol{\imath}} \frac{\partial}{\partial x} + \hat{\boldsymbol{\jmath}} \frac{\partial}{\partial y} + \hat{\boldsymbol{k}} \frac{\partial}{\partial z}$$

Therefore:

$$\vec{E} = -\left(\hat{\boldsymbol{\imath}} \frac{\partial V}{\partial x} + \hat{\boldsymbol{\jmath}} \frac{\partial V}{\partial y} + \hat{\boldsymbol{k}} \frac{\partial V}{\partial z} \right)$$

Since the voltage depends only on the z-coordinate, we can say: $V(\vec{r}) = V(z)$ so that $\frac{\partial V}{\partial x} = 0$ and $\frac{\partial V}{\partial y} = 0$.

$$\vec{E}(z) = -\hat{\boldsymbol{k}} \, \frac{\partial V}{\partial z}$$

$$\vec{E}(z) = -\hat{\boldsymbol{k}} \, \frac{\partial}{\partial z} \left(-\frac{\rho_0 z^3}{6K\epsilon_0} \right)$$

$$\vec{E}(z) = -\hat{\boldsymbol{k}} \, \left(-\frac{\rho_0 z^2}{2K\epsilon_0} \right)$$

$$\vec{E}(z) = \left(\frac{\rho_0 z^2}{2K\epsilon_0} \right) \, \hat{\boldsymbol{k}} \tag{2}$$

We see that Equation (1) and Equation (2) match each other as expected.

To get the charge density function we again use Gauss' Law:

$$\text{Flux } = 2AE(z) = \frac{Q_{enclosed}}{\epsilon_0}$$

We re-arrange to get at density function:

$$Q_{enclosed} = 2A\epsilon_0 E(z)$$

$$\int dQ = 2A\epsilon_0 E(z)$$

$$\int \rho(z) d\mathcal{V} = 2A\epsilon_0 E(z)$$

Now we take advantage of symmetry argument:

$$2 \int_{z'=0}^{z'=z} \rho(z') A \, dz' = 2A\epsilon_0 E(z)$$

$$\int_{z'=0}^{z'=z} \rho(z') \, dz' = \epsilon_0 E(z)$$

$$\int_{z'=0}^{z'=z} \rho(z') \, dz' = \epsilon_0 \left(\frac{\rho_0 z^2}{2K\epsilon_0} \right)$$

$$\int_{z'=0}^{z'=z} \rho(z')\, dz' = \left(\frac{\rho_0 z^2}{2K}\right)$$

Now we take advantage of the calculus rule that says that the derivative of an integral of a function is just that function:

$$\frac{d}{dx}\int_{x'=0}^{x'=x} f(x')\, dx' = f(x)$$

Therefore, we apply this identity after taking the derivative of boths sides of the previous equation:

$$\frac{d}{dz}\left[\int_{z'=0}^{z'=z} \rho(z')\, dz'\right] = \frac{d}{dz}\left[\left(\frac{\rho_0 z^2}{2K}\right)\right]$$

$$\rho(z) = \frac{\rho_0 z}{K}$$

$$\boxed{\rho(z) = \rho_0 \left(\frac{z}{K}\right)}$$

Just as we expect!

Part (d)

$\boxed{\text{The "student" is wrong.}}$ Gauss' Law says:

$$\int_{closedsurface} \vec{E}\cdot d\vec{A} = \frac{Q_{enclosed}}{\epsilon_0}$$

Since the charge distribution function ρz is **always positive** for any value of z then it must be true that the enclosed charge is *always positive*.

This means that by **Gauss' Law** the flux through the surface must be *always positive* for end-caps at any value of z.

Flux positive means that the field lines go from *inside* to *outside*. That is radially away from $z = 0$.

Moving any test charge in the same direction as the field line means moving in the direction of the electric force, and therefore doing positive work. This means that the potential energy is *decreasing* and therefore the voltage must **go down** as a function of distance. Indeed, our solution shows that voltage is a negative number times a power of z so this means that with increasing z voltage indeed decreases.

The reason that the student is wrong is because the student forgets about the symmetry that is imposed on the problem. Yes, there is increasing positive charge as we moving upward along the positive z-axis. But there is also increasing positive charge as we move in the negative direction. Since there is a one-to-one correspondence between the positive charges on one size and the positive charges on the other the two effects cancel. The only "non-cancelling" charge is that "block" of charge within the range -z to +z which always gives rise to a field that points away from the z-axis.