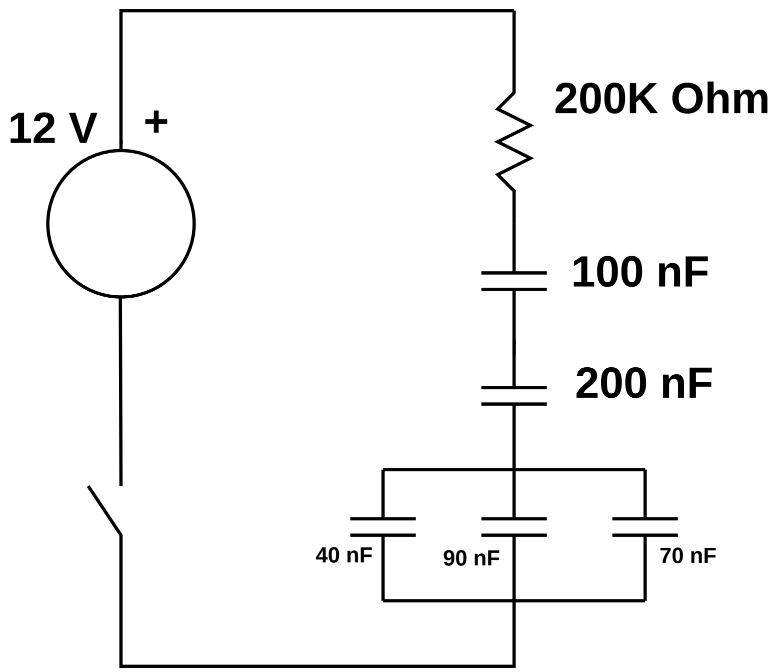
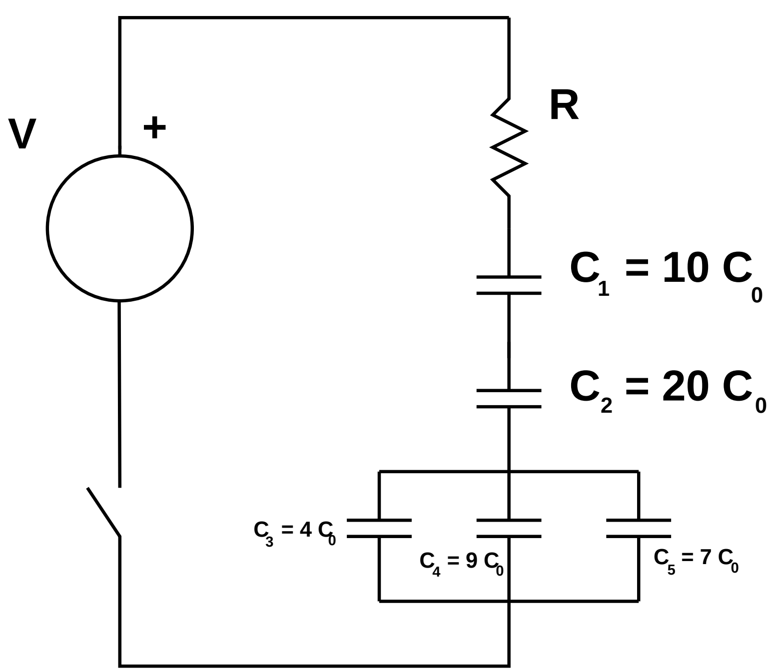
Solution to Practice Problem of the Day #18:

We are given this circuit with specific values for voltage and resistance:



To simplify our calculations and to avoid having to drag around units in our algebra, we convert the entire circuit to a simplified symbolic representation as shown:



Part (a): First, let's make clear from the onset **that the charge-up time-constant is the same for all five capacitors in this problem.** In other words, the time constant that applies to charging up C_1 is the same as the time constant for charging up C_3. This is because the current into the capacitor network is *not* governed by the capacitors. Rather, **the current is governed by the resistor**. In particular the current through the resistor will be proportial to the voltage drop across the resistor. Yes, that voltage drop depends on the voltage drop across the capacitor network, and so the capacitors do impact the time constant. But they impact the time constant as a network.

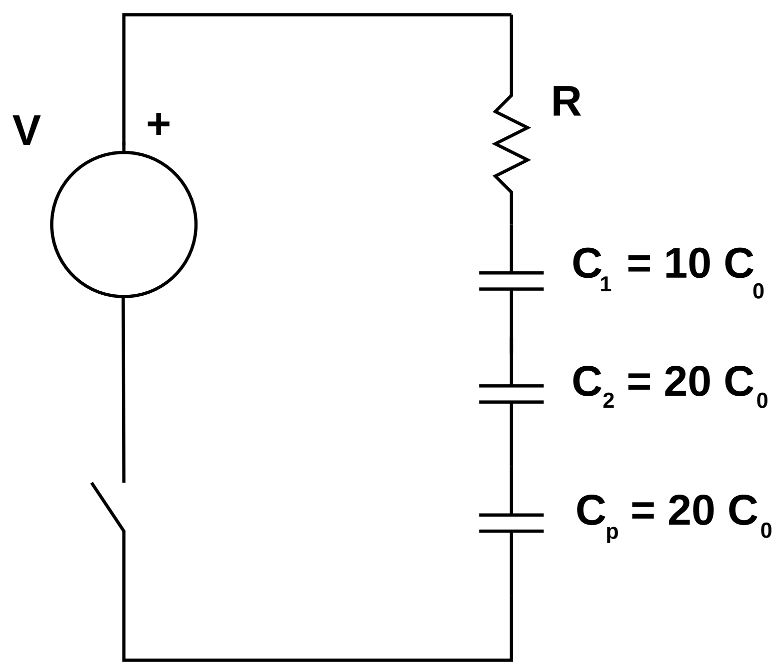
What this means is that (proportionally) all of the capacitors will charge up *together* as a result of charge being put *into* one end of the network and simultaneously taken *out of* the network on the other end. In other words, and any given time after the switch is closed we expect the ratio C_m/C_n to be a constant for any two of the five capacitors "n" and "m". (If this is not clear, we will show how this works exactly when we calculate the charges on each cap in part (b).

So there is only one time constant for the whole circuit and it has a value $\tau = RC_{eq}$ where C_{eq} is the **equivalent resistance** of the capacitor network.

We calculate the equivalent capacitance step-wise, starting with the three capacitors in parallel. We can replace these using the rule that says **caps in parallel add**:

$$C_p = C_3 + C_4 + C_5$$
$$C_p = 4C_0 + 9C_0 + 7C_0$$
$$C_p = 20C_0$$

Now we have three caps in series:



We use the rule that says **caps in series divide:**

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_p}$$
$$\frac{1}{C_{eq}} = \frac{1}{10C_0} + \frac{1}{20C_0} + \frac{1}{20C_0}$$

$$\frac{1}{C_{eq}} = \frac{2}{20C_0} + \frac{1}{20C_0} + \frac{1}{20C_0}$$
$$\frac{1}{C_{eq}} = \frac{4}{20C_0}$$
$$\frac{1}{C_{eq}} = \frac{1}{5C_0}$$
$$C_{eq} = 5C_0$$

Now we can calculate the time constant in terms of our symbolic variables:

$$\tau = RC_{eq}$$
$$\tau = R(5C_0)$$
$$\boxed{\tau = 5RC_0}$$

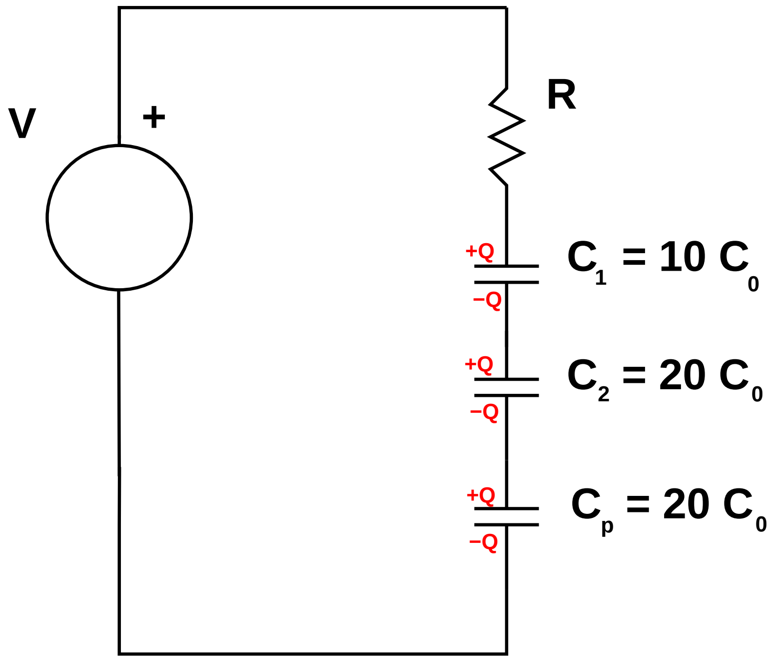
And we can plug in numbers as well, where R = 200K ohms = 200×10^3 ohms = 2×10^5 ohms, and where $C_0 = 10$ nanofarads = 10^{-8} farads:

$$\tau = 5(2 \times 10^5 \text{ ohms})(10^{-8} \text{ farads})$$
$$\boxed{\tau = 0.01 \text{ seconds}}$$

So we come to equilibrium in a fraction of a second.

Part (b):

To get the charges and the voltages we work the reverse of what we did to get the equivalent capacitance. First we consider the three capacitors in series again. Because the capacitors are in series, the charge on each capacitor must be the same value Q at any time – including when we get to equilibrium. Here is what it look like with the charges on each cap:



To get the voltages and the charges we apply **Kirchoff's Voltage Law** and note the following must be true:

$$V - \frac{Q}{C_1} - \frac{Q}{C_2} - \frac{Q}{C_p} = 0$$

In other words, when we go around our KVL loop and we go into capacitor on the "+Q" side and out the "-Q" side we drop the voltage given by $V_c = Q/C$. Note that the drop accross the resistor is zero because we are in equilibrium – no current, no voltage (Ohm's Law).

So we need to solve for Q. Some algebra:

$$Q\left(\frac{1}{C_1} - \frac{1}{C_2} - \frac{1}{C_p}\right) = V$$

Hey, we know this already:

$$Q\left(\frac{1}{C_{eq}}\right) = V$$

$$Q = C_{eq}V$$

This is a result which make intuitive sense if you think about it. In terms of symbolic variables then:

$$\boxed{Q_1 = Q_2 = Q = 5C_0V}$$

Plugging in numbers:

$$Q = (5 \times 10^{-8} \text{ farads})(12 \text{ volts})$$

$$Q = 6 \times 10^{-7} \text{ coulombs}$$

$$\boxed{Q = Q_1 = Q_2 = 600 \text{ nanocoulombs}}$$

Now to get the voltages we just use the voltage rule:

$$V_1 = \frac{Q_1}{C_1}$$

$$V_1 = \frac{5C_0V}{10C_0}$$

$$\boxed{V_1 = \frac{1}{2}V = 6 \text{ volts}}$$

Likewise:

$$V_2 = \frac{Q_2}{C_2}$$

$$V_2 = \frac{5C_0V}{20C_0}$$

$$\boxed{V_2 = \frac{1}{4}V = 3 \text{ volts}}$$

Since C_3, C_4, and C_5 are in parallel, they each experience the same voltage drop that the equivalent capacitor C_p would experience:

$$V_3 = V_4 = V_5 = V_p = \frac{Q_p}{C_p}$$

$$V_3 = V_4 = V_5 = V_p = \frac{5C_0V}{20C_0}$$

$$\boxed{V_3 = V_4 = V_5 = V_p = \frac{1}{4}V = 3 \text{ volts}}$$

Finally, we note that since the three capacitors are in parallel the charge that would correspond to the equivalent capacitance C_p is divided among the three capacitors in proportion to the capacitance:

$$Q_3 = \left(\frac{C_3}{C_p}\right) Q_p$$

$$Q_3 = \left(\frac{4C_0}{20C_0}\right) Q_p$$

$$Q_3 = \frac{1}{5} Q_p$$

$$Q_3 = \frac{1}{5}(5C_0V)$$

$$\boxed{Q_3 = C_0V = 120 \text{ nanocoulombs}}$$

Likewise:

$$Q_4 = \left(\frac{C_4}{C_p}\right) Q_p$$

$$Q_4 = \left(\frac{9C_0}{20C_0}\right) Q_p$$

$$Q_4 = \frac{9}{20} Q_p$$

$$Q_4 = \frac{9}{20}(5C_0V)$$

$$\boxed{Q_4 = \left(\frac{9}{4}\right) C_0V = 270 \text{ nanocoulombs}}$$

And finally:

$$Q_5 = \left(\frac{C_5}{C_p}\right) Q_p$$

$$Q_5 = \left(\frac{7C_0}{20C_0}\right) Q_p$$

$$Q_5 = \frac{7}{20} Q_p$$

$$Q_5 = \frac{7}{20}(5C_0V)$$

$$\boxed{Q_5 = \left(\frac{7}{4}\right) C_0 V = 210 \text{ nanocoulombs}}$$

Again, now that we have explicitly calculated the charge on each capacitor, it is clear that the charge on each of the five capacitors is always proportional to the charge Q corresponding to the charge put in and taken out.