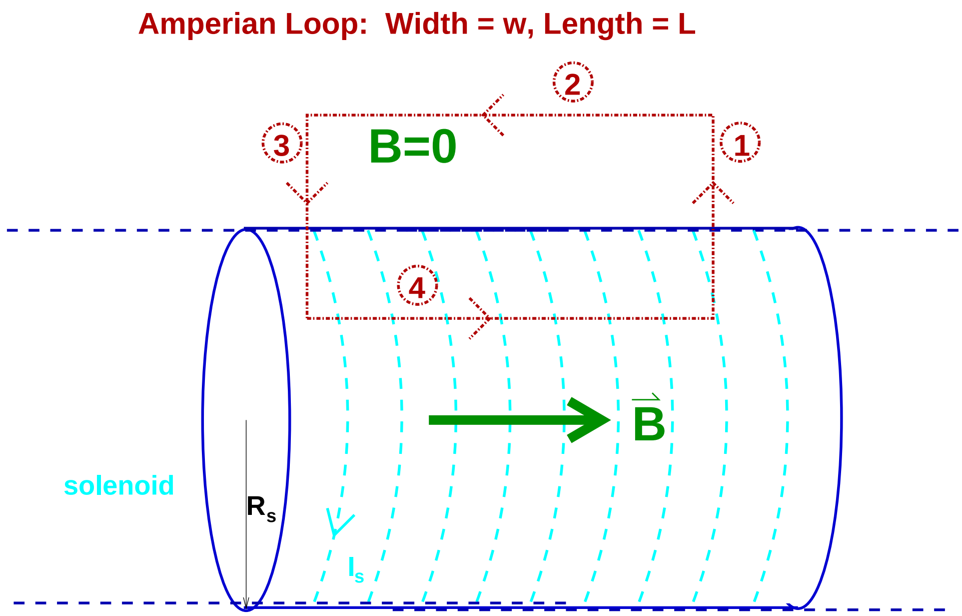
Solution to Practice Problem of the Day #16:

We write down Ampere's Law corresponding to this:

$$\oint_{loop} \vec{B} \cdot d\vec{s} = \mu_0 I_{enclosed}$$

The key (and non-obvious) step in this problem is defining a **Amperian Loop**. Given that the solenoid is axially symmetric, you might be tempted to draw an axial (circular) loop. This does not work, however. In fact, we need a Amperian loop that is at least (in part) lined up with the uniform field that we expect inside the solenoid. Such a loop is a **rectangle** of a given width w and length L which is partially embedded inside the solenoid.



We focus on the left side for starters. To calculate the path integral we can break the rectangular path into **four straight line segments** each labeled as shown. In other words:

$$\oint_{loop} \vec{B} \cdot d\vec{s} = \int_{①} \vec{B} \cdot d\vec{s} + \int_{②} \vec{B} \cdot d\vec{s} + \int_{③} \vec{B} \cdot d\vec{s} + \int_{④} \vec{B} \cdot d\vec{s}$$

Let's deal with these terms one at a time:

- For line segment ① we note that the path is *perpendicular* to the magnetic field on this segment. Therefore because of the dot-product, the path integral is zero:

$$\int_{①} \vec{B} \cdot d\vec{s} = 0$$

- Line segment ② lives in a region where the magnetic field is zero, Therefore, the path integral is zero:

$$\int_{②} \vec{B} \cdot d\vec{s} = 0$$

- For line segment ③ we note again, that the path is *perpendicular* to the magnetic field on this segment. Therefore because of the dot-product, the path integral is zero:

$$\int_{③} \vec{B} \cdot d\vec{s} = 0$$

Line segment ④ lives in a region where the magnetic field has a uniform value B. Furthermore the field is alinged with the path so the integral is quite easy:

$$\int_{④} \vec{B} \cdot d\vec{s} = \int_{④} B ds = B \int_{④} ds = B \int_0^L ds = BL$$

Therefore:

$$\oint_{loop} \vec{B} \cdot d\vec{s} = 0 + 0 + 0 + BL$$

$$\oint_{loop} \vec{B} \cdot d\vec{s} = BL$$

For the right hand side we consider the current enclosed within the rectangle. Each loop of solenoid containts current I_0. The density of loops in the sidewall of the solenoid is given as n and the number of enclosed loops is therefore $N = nL$ so the enclosed current is given by

$$I_{enclosed} = \mu_0 N I_0$$

$$I_{enclosed} = \mu_0 n L I_0$$

Putting this all together in Ampere's Law:

$$BL = \mu_0 n L I_0$$

$$\boxed{B = \mu_0 n I_0}$$