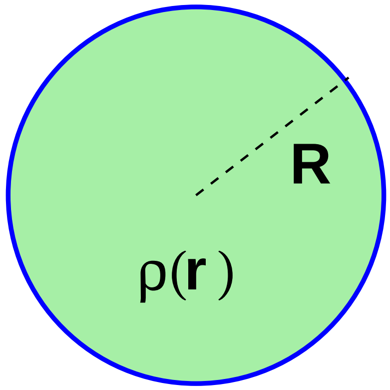
Solution to Practice Problem of the Day #14:

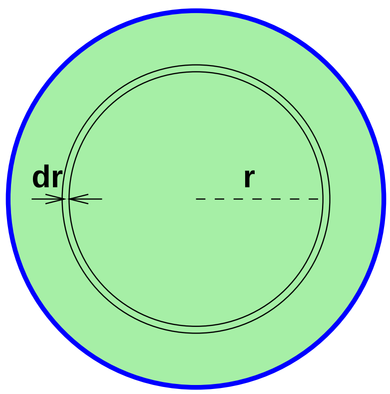


Part (a):

To get the total charge of the sphere we need to *integrate* the charge density over the volume of the sphere. The total charge in in some differential volume at a given radius corresponding to a particular value of the charge density is given by:

$$dq = \rho dV$$

where rho is the charge density at this particular radius and dV is the differential volume element. For this problem in particular, we want to define a *thin shell* of thickness dr at some radius $r < R$:



In this case the volume of the infinitesimally thin shell is just the thickness of the shell times the surface area:

$$dq = \rho(4\pi r^2)dr$$

Plugging in our expression for the density:

$$dq = \rho_0 \left(\frac{r}{R}\right)(4\pi r^2)dr$$

$$dq = \left(\frac{4\pi\rho_0}{R}\right) r^3 dr$$

To get the charge we integrate:

$$\int dq = \int \left(\frac{4\pi\rho_0}{R}\right) r^3 dr$$

$$Q = \left(\frac{4\pi\rho_0}{R}\right) \int_{r=0}^{r=R} r^3 dr$$

$$Q = \left(\frac{4\pi\rho_0}{R}\right) \frac{r^4}{4}\bigg|_{r=0}^{r=R}$$

$$\boxed{Q = \pi\rho_0 R^3}$$

It's interesting to note that in this case the total charge is precisely 3/4th what it would be if the density were uniform instead of linearly increasing.

Part (b):

To get the total charged enclosed within a sphere of radius r we just repeat the same integration only instead of integrating to a radius R we integrate to a radius r. To distinguish between the limit on the integral and the variable of integration, we use the "primed" notation:

$$Q(r) = \left(\frac{4\pi\rho_0}{R}\right) \int_{r'=0}^{r'=r} r'^3 dr'$$

$$Q(r) = \left(\frac{4\pi\rho_0}{R}\right) \frac{r'^4}{4}\bigg|_{r'=0}^{r'=r}$$

$$Q(r) = \left(\frac{4\pi\rho_0}{R}\right) \frac{r^4}{4}$$

$$\boxed{Q(r) = \left(\frac{\pi\rho_0 r^4}{R}\right)}$$

Note that the answer for Part (b) reduces to the answer for Part (a) for $r = R$ as expected.

Part (c):

For this part we apply **Gauss' Law**:

$$\Phi_E = \frac{Q_{enclosed}}{\epsilon_0}$$

$$\int_{\text{closed surface}} \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{\epsilon_0}$$

We consider the left-hand side first. This is the *flux* through a closed *Gaussian surface* with appropriate symmetry.

Since the charge distribution has **spherical symmetry**, we expect the electric field to depend only on the radius.

$$\vec{E}(\vec{r}) = \vec{E}(r) = E(r)\hat{r}$$

In other words, at at any given value of r, we can assume that the electric field has a particular magnitude and points outward from the center. This simplifies the surface integral calculation significantly:

$$\Phi_E = \int_{surface} [E(r)\hat{r}] \cdot d\vec{A}$$

Since $\hat{r}$ and $d\vec{A}$ point parallel to each other at every point on the Gaussian surface, the dot product gives us a scalar that is now a simply multiplication of the amplitudes:

$$\Phi_E = \int_{surface} E(r) dA$$

And we can pull out $E(r)$ since this is constant at a given r:

$$\Phi_E = E(r) \int_{surface} dA$$

And the integral of the differential area dA over a surface is just the surface area, which we know is $4\pi r^2$ for a sphere:

$$\Phi_E = E(r)(4\pi r^2)$$

So we put this into Gauss' Law and then we say for any spherical symmetry we can write:

$$E(r) = \left(\frac{1}{4\pi\epsilon_0}\right) \frac{Q_{enclosed}}{r^2}$$

It's worth mentioning that the steps above for Part (b) would be the same for *any* charge distribution with spherical symmetry.

Now we can look at the details of the charge distribution. There are two regions which for clarity we will define as follows:

- **Region I:** $r < R$,
- **Region II:** $r > R$,

For each of these we draw a spherical Gaussian surface and determine the charge enclosed:

- **Region I:** For $r < R$ we need an expression for the total charge enclosed within a radius r. Happily we already calculated this in Part (b):

$$Q_{I_{enclosed}}(r) = \left(\frac{\pi\rho_0 r^4}{R}\right)$$

Once we have the enclosed charge, we apply **Gauss' Law:**

$$E_I(r) = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{Q_{I_{enclosed}}(r)}{r^2}$$

$$E_I(r) = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{\left(\frac{\pi\rho_0 r^4}{R}\right)}{r^2}$$

$$E_I(r) = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{\pi\rho_0 r^2}{R}$$

$$\boxed{E_I(r) = \left(\frac{\rho_0 r^2}{4\epsilon_0 R}\right)}$$

- **Region II:** For $r > R$ We now enclose the entire charge of the shell, which we calculated in Part (a):

$$Q_{II_{enclosed}}(r) = \pi\rho_0 R^3$$

Once we have the enclosed charge, we apply **Gauss' Law:**

$$E_{II}(r) = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{Q_{II_{enclosed}}(r)}{r^2}$$

$$E_{II}(r) = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{\pi\rho_0 R^3}{r^2}$$

$$\boxed{E_{II}(r) = \left(\frac{\rho_0 R^3}{4\epsilon_0 r^2}\right)}$$

Part (d):

Once we have the electric field, all we need to do is the path integral:

$$V(\vec{r}) - V(\vec{r}_{ref}) = -\int_{\vec{r}_{ref}}^{\vec{r}} \vec{E}(\vec{s}) \cdot d\vec{s}$$

Here V is the electric potential (also called voltage), $\vec{r}_{ref}$ is the reference position and $\vec{r}$ is the position where we are trying to evaluate the voltage with respect to the reference.

In this problem, the electric field is radial and depends on r only. So the dot product simplifies and the path integral is just a regular 1-D integral:

$$V(r) - V(r_{ref}) = -\int_{r_{ref}}^{r} E(r')dr'$$

Note the critical importance of distinguishing between the variable of position r, and the variable of integration r'. We want to define the voltage as a function of position r. But to determine the voltage at position r we need to integrate the electric field from some reference position r_{ref} to r. To avoid confusion, we clearly label the variable of integration r'. We know we have done things correctly when in the end, all of the r' are gone, and we are only left with functions of r.

In this problem, we have selected the reference point as the *the point at infinity* $r = +\infty$ and we want to evaluate the voltage everywhere else. To do this we need to integrate **region-by-region** starting from the reference point.

For starters, we deal with **Region II** which is the first region we move through starting from the reference zero point at infinity:

$$V_{II}(r) - V(\infty) = -\int_{r'=+\infty}^{r'=r} E_{II}(r')dr'$$

$$V_{II}(r) - 0 = -\int_{r'=+\infty}^{r'=r} E_{II}(r')dr'$$

$$V_{II}(r) = -\int_{r'=+\infty}^{r'=r} E_{II}(r')dr'$$

$$V_{II}(r) = -\int_{r'=+\infty}^{r'=r} \left(\frac{\rho_0 R^3}{4\epsilon_0 r'^2}\right) dr'$$

$$V_{II}(r) = \left(\frac{\rho_0 R^3}{4\epsilon_0 r'}\right)\Bigg|_{r'=+\infty}^{r'=r}$$

$$\boxed{V_{II}(r) = \left(\frac{\rho_0 R^3}{4\epsilon_0 r}\right)}$$

In other words, just as we expect, in Region II, the voltage will fall as 1/r since the field looks like the field from a point source.

To get the voltage in **Region I** we need to integrate the electric field in this region, integrating from $r' = R$ to $r' = r$. First we work on the reference voltage:

$$V_I(r) - V_{II}(R) = -\int_{r'=R}^{r'=r} E_I(r')dr'$$

$$V_I(r) = V_{II}(R) - \int_{r'=R}^{r'=r} E_I(r')dr'$$

$$V_I(r) = \left(\frac{\rho_0 R^3}{4\epsilon_0 R}\right) - \int_{r'=R}^{r'=r} E_I(r')dr'$$

$$V_I(r) = \left(\frac{\rho_0 R^2}{4\epsilon_0}\right) - \int_{r'=R}^{r'=r} E_I(r')dr'$$

Next we solve the integral, using the results from Part (c) for $r < R$:

$$V_I(r) = \left(\frac{\rho_0 R^2}{4\epsilon_0}\right) - \int_{r'=R}^{r'=r} \left(\frac{\rho_0 r'^2}{4\epsilon_0 R}\right) dr'$$

$$V_I(r) = \left(\frac{\rho_0 R^2}{4\epsilon_0}\right) - \left(\frac{\rho_0 r'^3}{12\epsilon_0 R}\right)\Bigg|_{r'=R}^{r'=r}$$

$$V_I(r) = \left(\frac{\rho_0 R^2}{4\epsilon_0}\right) - \left(\frac{\rho_0 r^3}{12\epsilon_0 R}\right) + \left(\frac{\rho_0 R^2}{12\epsilon_0}\right)$$

$$V_I(r) = \left(\frac{3\rho_0 R^2}{12\epsilon_0}\right) - \left(\frac{\rho_0 r^3}{12\epsilon_0 R}\right) + \left(\frac{\rho_0 R^2}{12\epsilon_0}\right)$$

$$V_I(r) = \left(\frac{4\rho_0 R^2}{12\epsilon_0}\right) - \left(\frac{\rho_0 r^3}{12\epsilon_0 R}\right)$$

$$\boxed{V_I(r) = \left(\frac{\rho_0 R^2}{3\epsilon_0}\right) - \left(\frac{\rho_0 r^3}{12\epsilon_0 R}\right)}$$