Solution to Practice Problem of the Day #13:

Part (a):

This is a problem of applying **Gauss' Law**:

$$\Phi_E = \frac{Q_{enclosed}}{\epsilon_0}$$

$$\int_{\text{closed surface}} \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{\epsilon_0}$$

We consider the left-hand side first. This is the *flux* through a closed *Gaussian surface* with appropriate symmetry.

Since the charge distribution has spherical symmetry, we expect the electric field to depend only on the radius.

$$\vec{E}(\vec{r}) = \vec{E}(r) = E(r)\hat{r}$$

In other words, at at any given value of r, we can assume that the electric field has a particular magnitude and points outward from the center. This simplifies the surface integral calculation significantly:

$$\Phi_E = \int_{surface} [E(r)\hat{r}] \cdot d\vec{A}$$

Since $\hat{r}$ and $d\vec{A}$ point parallel to each other at every point on the Gaussian surface, the dot product gives us a scalar that is now a simply multiplication of the amplitudes:

$$\Phi_E = \int_{surface} E(r) dA$$

And we can pull out $E(r)$ since this is constant at a given r:

$$\Phi_E = E(r) \int_{surface} dA$$

And the integral of the differential area dA over a surface is just the surface area, which we know is $4\pi r^2$ for a sphere:

$$\Phi_E = E(r)(4\pi r^2)$$

So we put this into Gauss' Law and then we say for any spherical symmetry we can write:

$$E(r) = \left(\frac{1}{4\pi\epsilon_0}\right) \frac{Q_{enclosed}}{r^2}$$

(continues.....)

Now we can look at the details of the charge distribution. There are six regions which for clarity we will define as follows:

- **Region I:** $r < a$,
- **Region II:** $a < r < b$,
- **Region III:** $b < r < c$,
- **Region IV:** $c < r < d$,
- **Region V:** $d < r < e$ and
- **Region VI:** $r > e$.

For each of these we draw a spherical Gaussian surface and determine the charge enclosed:

- **Region I:** For $r < a$ we see immediately that there is nothing enclosed – no charge, or anything else for that matter. So the field is given by:

$$\boxed{E_I(r) = 0}$$

- **Region II:** For $a < r < b$ we enclose the thin conducting shell. But this is a "red herring" in that we know that the shell is *electrically neutral*. So therefore the enclosed charge is still zero and therefore we know that:

$$\boxed{E_{II}(r) = 0}$$

- **Region III:** This is the one region that requires some work. The *Gaussian Surface* at radius r for $b < r < c$ encloses some, but not all of the charge Q_i. Since the charge density is *uniform* the amount of charge is proportional to the fractional volume of the shell enclosed:

$$Q_{III_{enclosed}}(r) = Q_i \left(\frac{\text{Volume of insulating shell enclosed}}{\text{Volume of complete insulating shell}} \right)$$

The volume of a solid sphere of radius R is given by $\frac{4\pi}{3}R^3$. The volume of a shell with inner radius R_i and outer radius R_o is the volume of the sphere with radius R_0 minus the volume of the "hole" with radius R_i:

$$Q_{III_{enclosed}}(r) = Q_i \left(\frac{\frac{4\pi}{3}r^3 - \frac{4\pi}{3}b^3}{\frac{4\pi}{3}c^3 - \frac{4\pi}{3}b^3} \right)$$

therefore:

$$E_{III}(r) = \left(\frac{1}{4\pi r^2} \right) \frac{Q_i \left(\frac{r^3 - b^3}{c^3 - b^3} \right)}{\epsilon_0}$$

$$\boxed{E_{III}(r) = \left(\frac{Q_i}{4\pi\epsilon_0 r^2}\right)\left(\frac{r^3 - b^3}{c^3 - b^3}\right)}$$

- **Region IV:** For $c < r < d$ we are between the shells and so the enclosed charge is just the charge associated with the insulating shell:

$$Q_{IV_{enclosed}}(r) = Q_i$$

$$\boxed{E_{IV}(r) = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{Q_i}{r^2}}$$

- **Region V:** For $d < r < e$ we are *embedded in a perfect conductor* and therefore we automatically know that the field is zero:

$$\boxed{E_V(r) = 0}$$

- **Region VI:** For $r > e$ the Gaussian surface encloses both the insulating and the conducting shells. So the enclosed charge is just the sum of these two charges:

$$Q_{VI_{enclosed}}(r) = Q_i + Q_c$$

$$\boxed{E_{VI}(r) = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{Q_i + Q_c}{r^2}}$$

(continues....)

Part (b):

First we consider the charge induced on the *inner* surface of the conductor. We apply **Gauss' Law** for a Gaussian surface that is a sphere of radius r in **Region V** corresponding to the conductor:

$$\int_{\text{closed surface}} \vec{E_V} \cdot d\vec{A} = \frac{Q_{V_{enclosed}}}{\epsilon_0}$$

We know that in the conductor, **the field is automatically zero**. But Gauss's Law tells us that the enclosed charge is zero. The enclosed charge includes the charge of the insulting shell Q_i plus any charge induced on the inner surface of the conductor Q_{inner}. This has to total zero enclosed charge:

$$Q_i + Q_{inner} = 0$$

$$Q_{inner} = -Q_i$$

By symmetry, this charge should be uniformly spread across the surface. The surface charge density therefore is the total induced charge divided by the surface area of the inner surface of the conducting shell:

$$\sigma_{inner} = \frac{Q_{inner}}{4\pi R_{inner}^2}$$

$$\boxed{\sigma_{inner} = \frac{-Q_0}{4\pi d^2}}$$

For the outer surface of the conductor, we know that the net charge of the conducting shell has to be Q_c. So the surface charge has the value that satisfies this constraint, given our answer for the charge induced on the inner surface:

$$Q_{outer} + Q_{inner} = Q_c$$

$$Q_{outer} = Q_c - Q_{inner}$$

$$Q_{outer} = Q_c + Q_i$$

And again we divide this by the surface area to get the charge density:

$$\sigma_{outer} = \frac{Q_{outer}}{4\pi R_{outer}^2}$$

$$\boxed{\sigma_{outer} = \frac{Q_c + Q_i}{4\pi e^2}}$$

(continues....)

Part (c):

Once we have the electric field, all we need to do is the path integral:

$$V(\vec{r}) - V(\vec{r}_{ref}) = -\int_{\vec{r}_{ref}}^{\vec{r}} \vec{E}(\vec{s}) \cdot d\vec{s}$$

Here V is the electric potential (also called voltage), $\vec{r}_{ref}$ is the reference position and $\vec{r}$ is the position where we are trying to evaluate the voltage wrt the reference.

In this problem, the electric field is radial and depends on r only. So the dot product simplifies and the path integral is just a regular 1-D integral:

$$V(r) - V(r_{ref}) = -\int_{r_{ref}}^{r} E(r')dr'$$

Note the critical importance of distinguishing between the variable of position r, and the variable of integration r'. We want to define the voltage as a function of position r. But to determine the voltage at position r we need to integrate the electric field from some reference position r_{ref} to r. To avoid confusion, we clearly label the variable of integration r'. We know we have done things correctly when in the end, all of the r' are gone, and we are only left with functions of r.

In this problem, we have selected the reference point as the *inner surface of the conducting sphere* (corresponding to $r = d$) and we want to evaluate the voltage on the *outer surface of the insulating sphere* (corresponding to $r = c$). Happily, this means that we only need to deal with the field in one region (Region IV):

$$V(r = c) - V(r = d) = -\int_{r'=d}^{r'=c} E_{IV}(r')dr'$$

$$V(c) - V(d) = -\int_{r'=d}^{r'=c} E_{IV}(r')dr'$$

$$V(c) - V(d) = -\int_{r'=d}^{r'=c} \left(\frac{1}{4\pi\epsilon_0}\right) \frac{Q_i}{(r')^2}dr'$$

$$V(c) - V(d) = -\left(\frac{Q_i}{4\pi\epsilon_0}\right) \int_{r'=d}^{r'=c} \frac{1}{(r')^2}dr'$$

$$V(c) - V(d) = \left(\frac{Q_i}{4\pi\epsilon_0}\right) \frac{1}{r'}\Big|_{r'=d}^{r'=c}$$

$$\boxed{V(c) - V(d) = \left(\frac{Q_i}{4\pi\epsilon_0}\right)\left(\frac{1}{c} - \frac{1}{d}\right)}$$

Checking the sign, we note that the voltage changes in the positive direction, consistent with our expectation that moving a charged particle inward against the field will increase potential energy.

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Part (d):

Placing a large point charge in the outer-most region $r > e$ will certainly change the field in this particular region. The change will be non-symmetric and somewhat complicated (due to the requirement that the field lines intersect the conducting sphere at a right angle to the surface).

However, all other regions ($r < e$) lie withing the boundary of the perfectly conducting sphere. The conductor **shield** all inside regions from the effects of the external fields and charged.

Therefore $\boxed{\text{the field in Region IV}(r > e)\text{is changed}}$. However $\boxed{\text{the field in all other regions remains unchanged}}$.