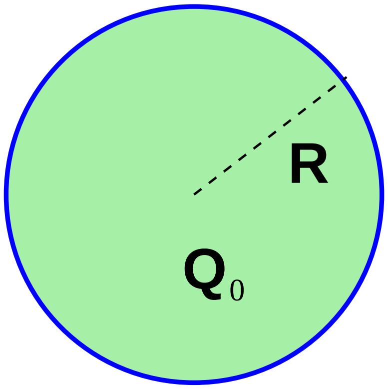
Solution to Practice Problem of the Day #12:



Part (a):

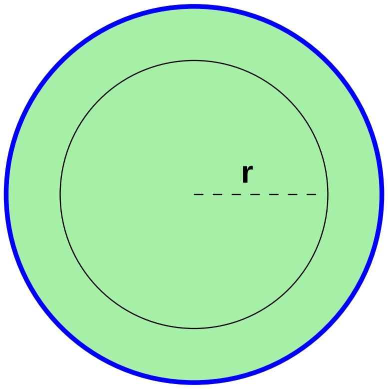
Since the density is *uniform* we can just calculate this directly by dividing the total charge by the total volume:

$$\rho(r < R) = \frac{\text{total charge}}{\text{total volume}} = \frac{Q_0}{\frac{4}{3}\pi R^3}$$

$$\boxed{\rho(r < R) = \frac{3Q_0}{4\pi R^3}}$$

Part (b):

To get the total charged enclosed within a sphere of radius r we note again that this is just the volume within this radius times the density which we calculate in Part (a). Alternatively, we can argue that the total charge within a sphere of radius r is just the total charge Q_0 times the fraction of the total sphere volume enclosed at radius r. We get the same answer either way:



$$Q_{enclosed}(r) = (\text{charge density})(\text{volume enclosed})$$

$$Q_{enclosed}(r) = \left(\frac{3Q_0}{4\pi R^3}\right)\left(\frac{4\pi r^3}{3}\right)$$

$$\boxed{Q_{enclose}(r) = Q_0 \left(\frac{r^3}{R^3}\right)}$$

Part (c):

For this part we apply **Gauss' Law**:

$$\Phi_E = \frac{Q_{enclosed}}{\epsilon_0}$$
$$\int_{\text{closed surface}} \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{\epsilon_0}$$

We consider the left-hand side first. This is the *flux* through a closed *Gaussian surface* with appropriate symmetry.

Since the charge distribution has **spherical symmetry**, we expect the electric field to depend only on the radius.

$$\vec{E}(\vec{r}) = \vec{E}(r) = E(r)\hat{r}$$

In other words, at at any given value of r, we can assume that the electric field has a particular magnitude and points outward from the center. This simplifies the surface integral calculation significantly:

$$\Phi_E = \int_{surface} [E(r)\hat{r}] \cdot d\vec{A}$$

Since $\hat{r}$ and $d\vec{A}$ point parallel to each other at every point on the Gaussian surface, the dot product gives us a scalar that is now a simply multiplication of the amplitudes:

$$\Phi_E = \int_{surface} E(r) dA$$

And we can pull out $E(r)$ since this is constant at a given r:

$$\Phi_E = E(r) \int_{surface} dA$$

And the integral of the differential area dA over a surface is just the surface area, which we know is $4\pi r^2$ for a sphere:

$$\Phi_E = E(r)(4\pi r^2)$$

So we put this into Gauss' Law and then we say for any spherical symmetry we can write:

$$E(r) = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{Q_{enclosed}}{r^2}$$

It's worth mentioning that the steps above for Part (b) would be the same for *any* charge distribution with spherical symmetry.

Now we can look at the details of the charge distribution. There are two regions which for clarity we will define as follows:

- **Region I:** $r < R$,
- **Region II:** $r > R$,

For each of these we draw a spherical Gaussian surface and determine the charge enclosed:

- **Region I:** For $r < R$ we need an expression for the total charge enclosed within a radius r. Happily we already calculated this in Part (b):

$$Q_{I_{enclosed}}(r) = Q_0 \left(\frac{r^3}{R^3}\right)$$

Once we have the enclosed charge, we apply **Gauss' Law:**

$$E_I(r) = \left(\frac{1}{4\pi\epsilon_0}\right) \frac{Q_{I_{enclosed}}(r)}{r^2}$$

$$E_I(r) = \left(\frac{1}{4\pi\epsilon_0}\right) \left(Q_0 \frac{r^3}{R^3}\right) \frac{1}{r^2}$$

$$E_I(r) = \left(\frac{1}{4\pi\epsilon_0}\right) Q_0 \frac{r}{R^3}$$

$$\boxed{E_I(r) = \frac{Q_0 r}{4\pi\epsilon_0 R^3}}$$

We note that the field fall *linearly* with distance then. Kind of interesting.

- **Region II:** For $r > R$ We now enclose the entire charge of the sphere which is just Q_0

$$Q_{II_{enclosed}}(r) = Q_0$$

Once we have the enclosed charge, we apply **Gauss' Law:**

$$E_{II}(r) = \left(\frac{1}{4\pi\epsilon_0}\right) \frac{Q_{II_{enclosed}}(r)}{r^2}$$

$$E_{II}(r) = \left(\frac{1}{4\pi\epsilon_0}\right) \frac{Q_0}{r^2}$$

$$\boxed{E_{II}(r) = \frac{Q_0}{4\pi\epsilon_0 r^2}}$$

This is precisely the same field as for a point charge Q_0 which is just what we expect.

Part (d):

Once we have the electric field, all we need to do is the path integral:

$$\Delta V = V(\vec{r}) - V(\vec{r}_{ref}) = -\int_{\vec{r}_{ref}}^{\vec{r}} \vec{E}(\vec{s}) \cdot d\vec{s}$$

Here V is the electric potential (also called voltage), $\vec{r}_{ref}$ is the reference position and $\vec{r}$ is the position where we are trying to evaluate the voltage with respect to the reference.

In this problem, the electric field is radial and depends on r only. So the dot product simplifies and the path integral is just a regular 1-D integral:

$$\Delta V = V(r) - V(r_{ref}) = -\int_{r_{ref}}^{r} E(r')dr'$$

Note the critical importance of distinguishing between the variable of position r, and the variable of integration r'. We want to define the voltage as a function of position r. But to determine the voltage at position r we need to integrate the electric field from some reference position r_{ref} to r. To avoid confusion, we clearly label the variable of integration r'. We know we have done things correctly when in the end, all of the r' are gone, and we are only left with functions of r.

In this problem, we have selected the reference point as the *the point at infinity* $r = +\infty$ and we want to evaluate the voltage everywhere else. To do this we need to integrate **region-by-region** starting from the reference point.

For starters, we deal with **Region II** which is the first region we move through starting from the reference zero point at infinity:

$$V_{II}(r) - V(\infty) = -\int_{r'=+\infty}^{r'=r} E_{II}(r')dr'$$

$$V_{II}(r) - 0 = -\int_{r'=+\infty}^{r'=r} E_{II}(r')dr'$$

$$V_{II}(r) = -\int_{r'=+\infty}^{r'=r} E_{II}(r')dr'$$

Now we plug in the expression electric field:

$$V_{II}(r) = -\int_{r'=+\infty}^{r'=r} \left(\frac{Q_0}{4\pi\epsilon_0 r'^2}\right) dr'$$

$$V_{II}(r) = \left(\frac{Q_0}{4\pi\epsilon_0 r'}\right)\Big|_{r'=+\infty}^{r'=r}$$

$$\boxed{V_{II}(r) = \left(\frac{Q_0}{4\pi\epsilon_0 r}\right)}$$

In other words, just as we expect, in Region II, the voltage will fall as 1/r since the field looks like the field from a point source.

To get the voltage in **Region I** we need to integrate the electric field in this region, integrating from $r' = R$ to $r' = r$. The voltage at the surface of the sphere effective constitutes a new "reference" point – not the zero point, but rather the point from which we can integrate a particular function. Since the voltage function must be continuous, we use the value at the boundary that is determined from the definition of the voltage in Region II.

$$V_I(r) - V_{II}(R) = -\int_{r'=R}^{r'=r} E_I(r')dr'$$

$$V_I(r) = V_{II}(R) - \int_{r'=R}^{r'=r} E_I(r')dr'$$

$$V_I(r) = \left(\frac{Q_0}{4\pi\epsilon_0 R}\right) - \int_{r'=R}^{r'=r} E_I(r')dr'$$

Next we solve the integral, using the results from Part (c) for $r < R$:

$$V_I(r) = \left(\frac{Q_0}{4\pi\epsilon_0 R}\right) - \int_{r'=R}^{r'=r} \left(\frac{Q_0 r'}{4\pi\epsilon_0 R^3}\right) dr'$$

$$V_I(r) = \left(\frac{Q_0}{4\pi\epsilon_0 R}\right) - \left(\frac{Q_0 r'^2}{8\pi\epsilon_0 R^3}\right)\Big|_{r'=R}^{r'=r}$$

$$V_I(r) = \left(\frac{Q_0}{4\pi\epsilon_0 R}\right) - \left(\frac{Q_0 r^2}{8\pi\epsilon_0 R^3}\right) + \left(\frac{Q_0 R^2}{8\pi\epsilon_0 R^3}\right)$$

$$\boxed{V_I(r) = \left(\frac{Q_0}{4\pi\epsilon_0 R}\right) + \left(\frac{Q_0 R^2}{8\pi\epsilon_0 R^3}\right) - \left(\frac{Q_0 r^2}{8\pi\epsilon_0 R^3}\right)}$$

Or if you prefer a slightly more symmetrical forms:

$$\boxed{V_I(r) = \left(\frac{3Q_0 R^2}{8\pi\epsilon_0 R^3}\right) - \left(\frac{Q_0 r^2}{8\pi\epsilon_0 R^3}\right)}$$

$$\boxed{V_I(r) = \left(\frac{Q_0}{8\pi\epsilon_0 R^3}\right)\left(3R^2 - r^2\right)}$$