Solution to Practice Problem of the Day #11:

The goal here is to directly integrate the differential form of Coulomb's Law:

$$\vec{E} = \int_{allspace} d\vec{E}$$

where

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2}\hat{r}$$

There are three steps to take before we even consider applying the integral sign:

- **Step 1:** We need to "reparameterize" the differential dq
- **Step 2:** We need to deal with the radius term r^2
- **Step 3:** We need to deal with the unit vector $\hat{r}$

Only after we have done these things can we actually start to tackle the integral itself.

Step 1: In general for a 3-D distribution we need to specify the differential charge in terms of a volume element in three physical coordinates. For example, if we were given a solid shape with a unform volume charge density ρ then we could say:

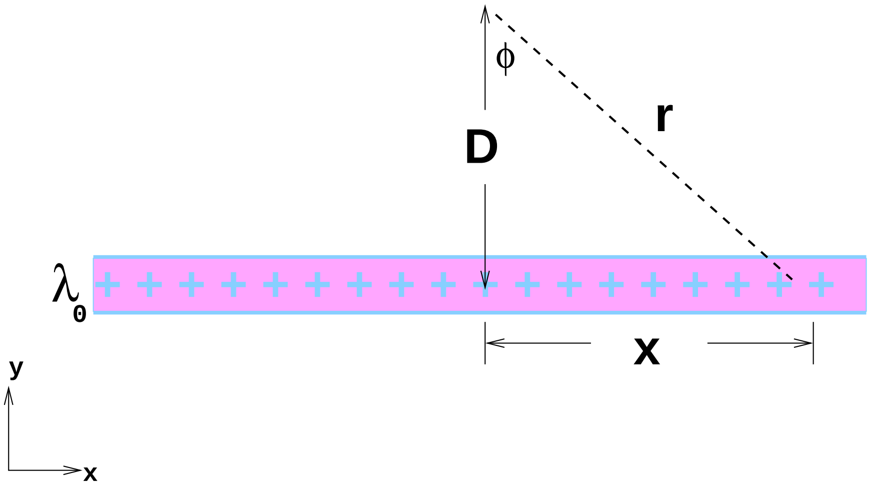
$$dq = \rho\, dx\, dy\, dz$$

However, in this problem we do not have a volume, we have a "line of charge". That is to say, we have a thin line of charge that has a uniform charge density (units Coulombs per meter). Since we have a "1-D" line of charge, we only need to specify the differential charge in terms of this one coordinate

$$dq = \lambda_0\, ds$$

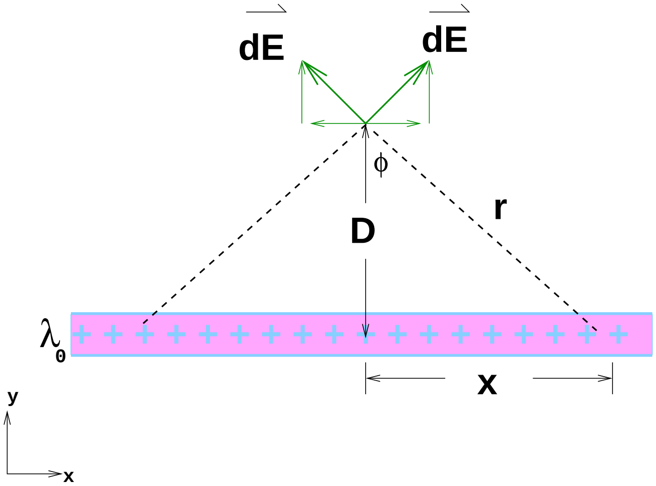
where ds is a differential step along the coordinate of the charge line. For this particular problem the charge line is aligned with the "x" coordinate so: $ds = dx$. Therefore:

$$dq = \lambda_0 dx$$



Step 2: Now we need to deal with the radius. Specifically, if we call the distance from the line D then the length of r is not constant as we move along x. In particular, we see that there is a "right triangle" so we use the Pythagorean Theorem above:[1]

$$r^2 = D^2 + x^2$$



Step 3: As can be seen in the figure above, the mirror symmetry of the charge distribution provides a handy way to simplify our consideration of the different components of $\hat{r}$. Specifically, we can see that for every piece of charge on the left side of $x = 0$, there is a equal piece of charge on the right side. Each charge bit contributes a differential force pointed radially at the point. The total field at the point is the vector sum of the field from the two charge bits. We see that the horizontal components of the summed fields cancel, leaving only the "vertical" component. In other words only the component in the y-direction. So we only need evaluate dE_y. We can also see from the figure that this component is given by:

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[1] For this solution I use capital "D" instead of lower-case "d" to avoid possible confusion with the "derivative" symbol.

$$dE_y = dE \cdot \cos(\phi)$$

Putting all three steps together, we have:

$$dE_y = \left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{\lambda_0 dx}{x^2 + D^2}\right)\cos(\phi)$$

Finally, we are ready to to integrate this:

$$E_y = \int_{-\infty}^{+\infty}\left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{\lambda_0 dx}{x^2 + D^2}\right)\cos(\phi)$$

Any standard method of integration is acceptable. Annoyingly, ϕ depends on x and vice versa, so you need to convert to do the integral in one or the other coordinate. Also annoying is the fact that the integrand x runs from negative to positive infinity, which is "inconvenient" when it comes to doing algebra. Still, there are several ways to "solve" this integral by parameterizing the integrand in terms of either x or ϕ. But solving this integral is not physics – it's calculus. Here's my solution, using a trig substitution to parameterize in terms of ϕ only:

- We recognize that we are integrating an "even" function. That is to say, the integral from negative to positive infinity is twice the value from 0 to positive infinity since the function is a "mirror image" about $x = 0$. So:

$$E_y = \int_{0}^{+\infty}\left(\frac{2}{4\pi\epsilon_0}\right)\left(\frac{\lambda_0 dx}{x^2 + D^2}\right)\cos(\phi)$$

- We do a "trig substitution" to do this integral. We change from a linear integrand (dx) to an angular integrand ($d\phi$):

$$x = D\tan(\phi)$$

$$dx = D(\sec\phi)^2 d\phi$$

We plug this in, and adjust the integral which used to run from $x = 0$ to $x = +\infty$ and which now runs from $\phi = 0$ to $\phi = \frac{+\pi}{2}$.

Pulling out some constants:

$$E_y = \left(\frac{2\lambda_0}{4\pi\epsilon_0 D}\right)\int_0^{\frac{\pi}{2}}\left[\frac{(\sec\phi)^2 d\phi}{(\tan\phi)^2 + 1}\right]\cos(\phi)$$

Multiply the fraction inside the integral top and bottom by $(cos\phi)^2$:

$$E_y = \left(\frac{2\lambda_0}{4\pi\epsilon_0 D}\right)\int_0^{\frac{\pi}{2}}\left[\frac{d\phi}{(\sin\phi)^2 + (\cos\phi)^2}\right]\cos(\phi)$$

Make use the the trig identity that says sine-squared plus cosine-squared is always equal to one:

$$E_y = \left(\frac{2\lambda_0}{4\pi\epsilon_0 D}\right)\int_0^{\frac{\pi}{2}} \cos(\phi) d\phi$$

Wow, now this is a *really* easy integral:

$$E_y = \left(\frac{2\lambda_0}{4\pi\epsilon_0 D}\right)\sin(\phi)\Big|_0^{\frac{\pi}{2}}$$

$$E_y = \frac{2\lambda_0}{4\pi\epsilon_0 D}$$

$$\boxed{E_y = \frac{\lambda_0}{2\pi\epsilon_0 D}}$$

Q.E.D. (Quite Easily Done.)