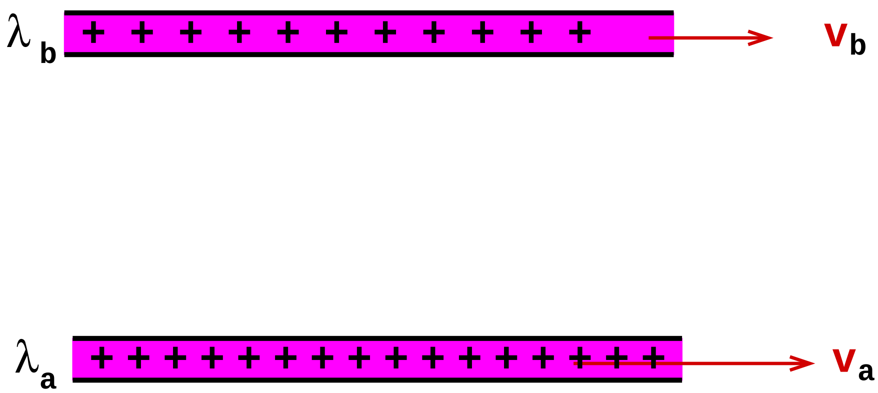
Solution to Practice Problem of the Day #09:

We start by modeling each conducting wire as the superposition of two linear charge densities, one λ_+ corresponding to the linear charged density of positive charges, and one λ_- corresponding to the linear charged density of negative charges. We argue that the condition that the wire be electrically neutral means $\lambda_- = -\lambda_+$. We can create a non-zero current by allowing the positive charges to move freely through the wire whilst the negative charges stay put.[1]

We therefore set up two infinite wires, each parallel to each other. The lower wire has moving positive linear charge density λ_a moving to the right at speed v_a and the upper wires has positive linear charge density λ_b moving to the right at speed v_b.



We note that for each wire, we can easily connect the values of the charge densities and charge speeds with the current. For example, in Wire "a" we can apply the "chain rule" of calculus to say:

$$I_a = \frac{dq}{dt} = \frac{dq}{dx}\frac{dx}{dt} = \lambda_a v_a$$

And likewise for Wire "b":

$$I_b = \frac{dq}{dt} = \frac{dq}{dx}\frac{dx}{dt} = \lambda_b v_b$$

In other words, the current (charge per unit time) is just the charge density (charge per unit length) multiplied by the velocity of the charges (length per unit time).

The magnetic field due to a wire is given by Biot-Savart for a wire:

$$B_{wire} = \frac{\mu_0 I}{2\pi r}$$

We consider the field on Wire "b" due to Wire "a": Here $I_a = \lambda_a v_a$ and $r = d$ corresponding to the distance between the two wires. Therefore the field *at* Wire "b", *due to* Wire "a" is:

[1] Yes, in real wires, the opposite hold: the negative electrons move whilst the positive ions stay put. But we have already argued that a current of negative charges in one direction is exactly equivalent to a current of positive charges in the opposite direction.

$$B_{ba} = \frac{\mu_0 \lambda_a v_a}{2\pi d}$$

Now we consider the force that results on the charges in Wire "b". We know that:

$$\vec{F} = q\vec{v} \times \vec{B}$$

We note that $\vec{v}_b$ is perpendicular to $\vec{B}_{ba}$, and so we can quickly calculate the magnitude of the force on each charge:

$$|\vec{F}| = q_b v_b B_{ba}$$

Now to get the force on a length of wire with length ℓ we note that the total positive charge on the length of wire is just the length times the linear charge density:

$$q_b = \lambda_b \ell$$

Plugging this in:

$$F_{ba} = \lambda_b \ell v_b B_{ba}$$

Putting in the expression for the B-field:

$$F_{ba} = \lambda_b \ell v_b \frac{\mu_0 \lambda_a v_a}{2\pi d}$$

Rearranging:

$$F_{ba} = \frac{\mu_0 (\lambda_a v_a)(\lambda_b \ell v_b)}{2\pi d}$$

And using our expression for current in terms of velocity and charge densities:

$$\boxed{F_{ba} = \frac{\mu_0 I_a I_b}{2\pi d}}$$