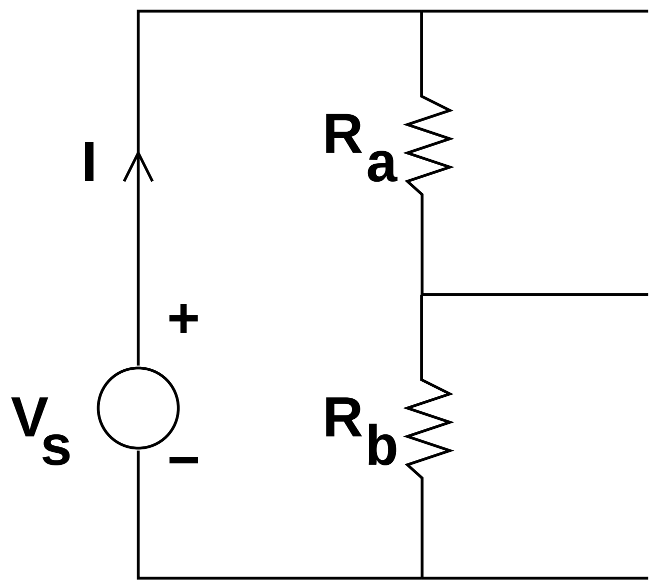
Solution to Practice Problem of the Day #07:

Part (a)

If we have **ideal voltmeters** then these are devices for measuring voltage that appears at the positive terminal relative to the negative terminal. In particular, **the ideal voltmeter accepts zero current**. Where the voltmeter is corresponds to an "open circuit" – a switch in the "off" position:



So we see immediately that $\mathcal{V}_1$ corresponds to the voltage drop accross resistor R_a and $\mathcal{V}_2$ corresponds to the voltage drop accross resistor R_b.

First, we note that this is now a simple loop with two resistors in series. We use the **series current rule** which says that the current in each component must be the same:

$$I_a = I_b \equiv I$$

Likewise we can see that the voltage across both resistors together must be the voltage applied by the **voltage source**:

$$\mathcal{V}_s = \mathcal{V}_1 + \mathcal{V}_2$$

Now we are ready to apply **Ohm's Law:** to solve for the current:

$$\mathcal{V}_s = I_a R_a + I_b R_b$$
$$\mathcal{V}_s = I R_a + I R_b$$
$$\mathcal{V}_s = I(R_a + R_b)$$
$$I = \frac{\mathcal{V}_s}{R_a + R_b}$$

To get the voltages, we again consider Ohm's Law one resistor at a time:

$$\mathcal{V}_1 = I_a R_a$$
$$\mathcal{V}_1 = \left(\frac{\mathcal{V}_s}{R_a + R_b}\right) R_a$$
$$\boxed{\mathcal{V}_1 = \mathcal{V}_s \left(\frac{R_a}{R_a + R_b}\right)}$$

Likewise:

$$\mathcal{V}_2 = I_b R_b$$
$$\mathcal{V}_2 = \left(\frac{\mathcal{V}_s}{R_a + R_b}\right) R_b$$
$$\boxed{\mathcal{V}_2 = \mathcal{V}_s \left(\frac{R_b}{R_a + R_b}\right)}$$

Part (b)

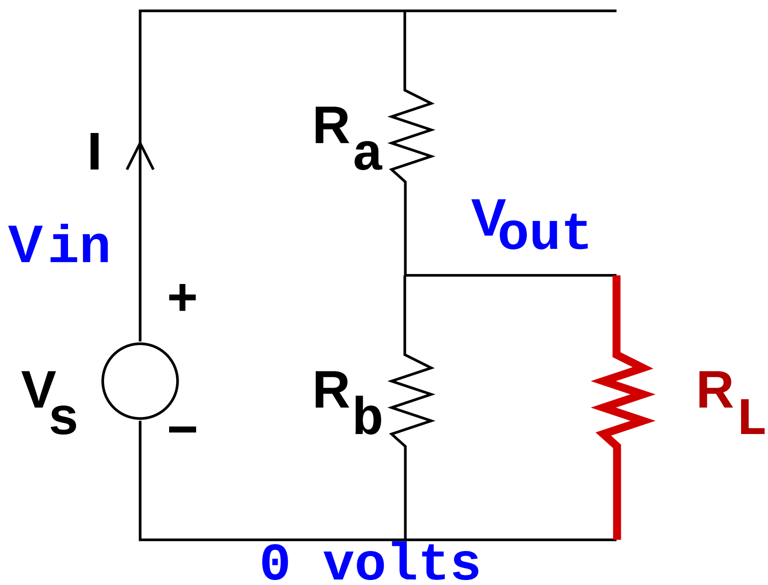
This kind of circuit where we have a known voltage applied accross two resistors in series is called a **Voltage Divider**. The current is the same in every resistor in series. But the voltage is not. In fact, the voltage is *divided* between the two resistors, with the sum of the two voltage drops being equal to the total applied voltage from the source.

By the same logic we might call two resistors in parallel (where the voltage drop is the same for each resistor) a "current divider" because the total current coming into the circuit is *divided* between the different arms.

A **voltage divider** is a useful circuit contruct because it allows you to generate a any desired lower voltage from any voltage source which runs at a higher voltage by simply chosing resistor values. For example, if you had a 24 volt source but you wanted to generate a 12-volt output, you could use two identical resistors to do this. The main advantage of a voltage divider is that it is a simple and easy way to generate some desired voltage. The main drawbacks are that they use power (dissappated into the resistors) and they fail if the output voltage is attached to a load that draws current (see part c).

Part (c)

Putting a "load" on our voltage divider is represented by the figure below:



The equivalent resistance of the load is represented by R_L. The voltage divider is designed to act as a "function generator" where the output voltage (corresponding to $\mathcal{V}_2$) is held at a fixed value proportional to the input (source) voltage (corresponding to $\mathcal{V}_s$):

$$\mathcal{V}_{out} = \mathcal{V}_{in}\left(\frac{R_b}{R_a + R_b}\right)$$

This is the *design* output of the voltage divider. In fact, $\mathcal{V}_{out}$ will match the design output in the limit where the load resistance is infinite. However, if the load resistance has a finite value, then some of the current that goes through the upper resistor will go through the load instead of the lower resistor. Since there is less current in the lower resistor, the output of the divider $\mathcal{V}_{out}$ will be *less* than the design output. In other words the voltage divider *sags* if the load resistance is low. We can work out how much sagging occurs by solving for the actual voltage V_{out} with a finite load:

First, we look at the **parallel structure**:

$$\mathcal{V}_{out} = \mathcal{V}_b = \mathcal{V}_L$$

Applying **Ohm's Law:**

$$\mathcal{V}_{out} = I_b R_b = I_L R_L$$

Next let's look again at the upper resistor:

$$V_a = I_a R_a$$

Adding up voltages using our voltage divider naming scheme:

$$V_{in} = V_a + V_{out}$$

And now by **conservation of current** we have:

$$I_a = I_b + I_L$$

Using our previous expressions and plugging in for these currents:

$$\frac{V_a}{R_a} = \frac{V_{out}}{R_b} + \frac{V_{out}}{R_L}$$

$$\frac{V_{in} - V_{out}}{R_a} = \frac{V_{out}}{R_b} + \frac{V_{out}}{R_L}$$

$$\frac{V_{in}}{R_a} = \frac{V_{out}}{R_a} + \frac{V_{out}}{R_b} + \frac{V_{out}}{R_L}$$

$$V_{in} = V_{out} R_a \left(\frac{1}{R_a} + \frac{1}{R_b} + \frac{1}{R_L} \right)$$

$$V_{in} = V_{out} R_a \left(\frac{R_b R_L}{R_a R_b R_L} + \frac{R_a R_L}{R_a R_b R_L} + \frac{R_a R_b}{R_a R_b R_L} \right)$$

$$V_{in} = V_{out} \left(\frac{R_b R_L + R_a R_L + R_a R_b}{R_b R_L} \right)$$

$$V_{out} = V_{in} \left(\frac{R_b R_L}{R_b R_L + R_a R_L + R_a R_b} \right)$$

In the limit where $R_L \gg R_a$ and $R_L \gg R_b$ we can say that the third term in the denominator is small compared to the other two terms. In this case, the answer reduces to our previous answer for Part (a).

$$V_{out} \approx V_{in} \left(\frac{R_b R_L}{R_b R_L + R_a R_L + 0} \right)$$

$$V_{out} \approx V_{in} \left(\frac{R_b R_L}{R_L (R_b + R_a)} \right)$$

$$\boxed{V_{out} \approx V_{in} \left(\frac{R_b}{R_b + R_a} \right)}$$

In contrast if $R_L = R_a = R_b \equiv R$ then

$$V_{out} = V_{in} \left(\frac{R^2}{3R^2} \right)$$

$$V_{out} = \frac{1}{3} V_{in}$$
$$V_{out} = \frac{1}{3} V_{in}$$

In other words, the actual output voltage one-third of the input voltage instead of the *design output of one-half of the input voltage*:

$$V_{out}(\text{design}) = \frac{V_{in}}{2}$$
$$V_{out}(\text{actual}) = \frac{V_{in}}{3}$$

This corresponds to a performanace of about 67% of the desired voltage, corresponding to **a sag of 33% relative to the design value.**